# Concurrency Theory

## Lecture 17: Interleaving Semantics of Petri Nets

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http://moves.rwth-aachen.de/teaching/ws-1718/ct

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## Overview







## Overview







## Carl Adam Petri (1926-2010)

The original work[1] does not contain a single (graphical) Petri net!

[1] Petri's PhD dissertation, 1962.

# Semantics: executions and traces

Models in the 60s: lambda calculus, finite automata, Turing machines, ...

States: current configurations of the machine

One or more initial states

Possibly some distinguished final states

Transitions: moves between configurations

| | | | |
|---|---|---|---|
| Lambda calculus | $(\lambda x.xx)(\lambda y.y)$ | $\longrightarrow$ | $(\lambda y.y)(\lambda z.z)$ |
| Turing machine | $0010q_1011$ | $\longrightarrow$ | $001q_201011$ |
| Finite automaton | $q_1$ | $\xrightarrow{a}$ | $q_2$ |
| Pushdown automaton | $(q_1, XYYZ)$ | $\xrightarrow{a}$ | $(q_2, XYXYYZ)$ |

Executions: alternating sequences of states and transitions





# Petri's question

C.A. Petri points out a discrepancy between how Theoretical Physics and Theoretical Computer Science described systems in 1962:

Theoretical Physics describes systems as a collection of interacting particles (subsystems), without a notion of global clock or simultaneity

Theoretical Computer Science describes systems as sequential virtual machines going through a temporally ordered sequence of global states

Petri's question:

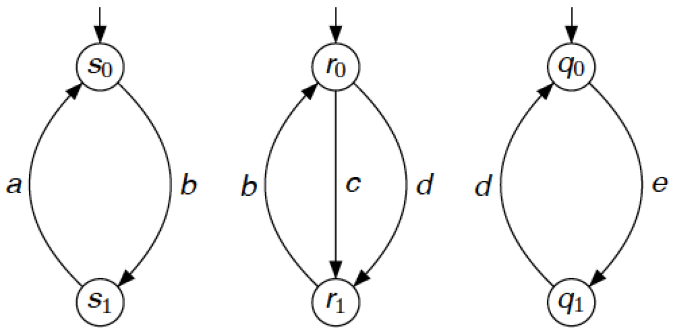
Which kind of abstract machine should be used to describe the physical implementation of a Turing machine?





# Petri net

A graphical representation of interacting finite automata:





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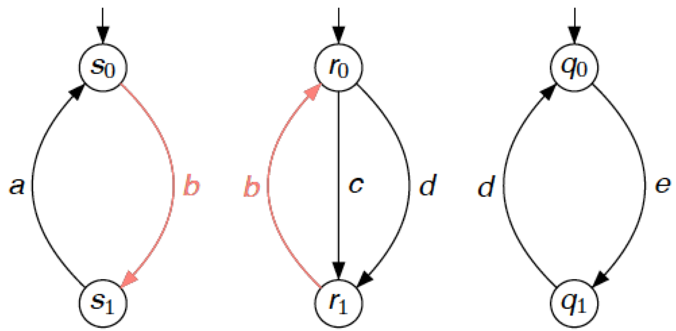

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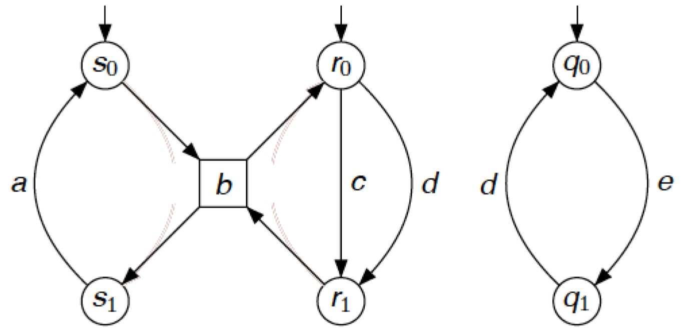






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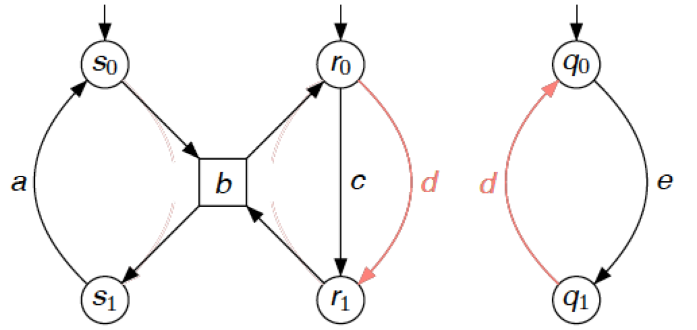






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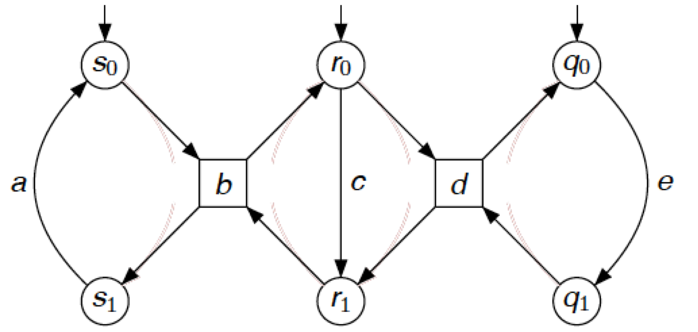






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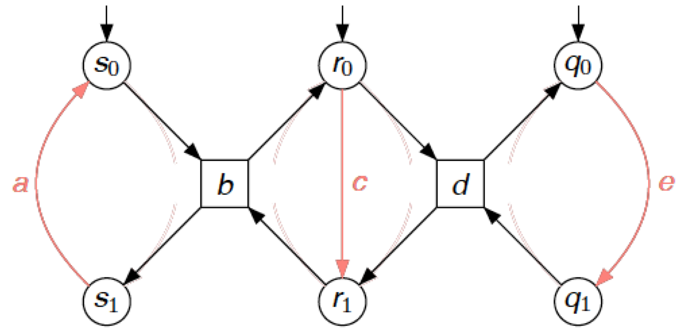

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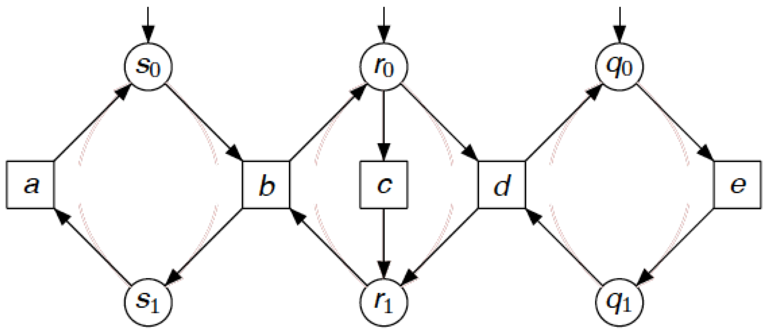
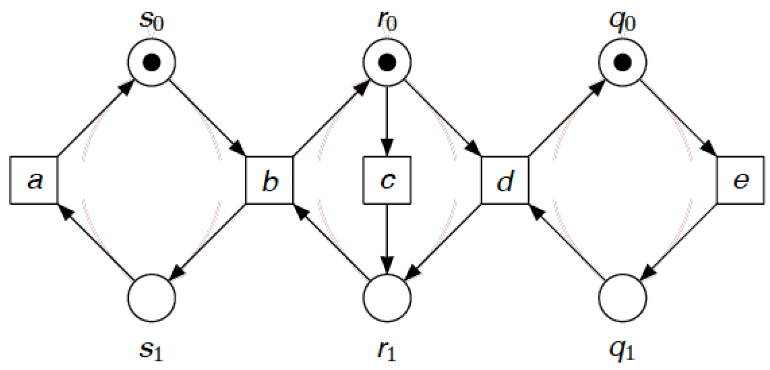







## Petri net

A graphical representation of interacting finite automata:








## Overview







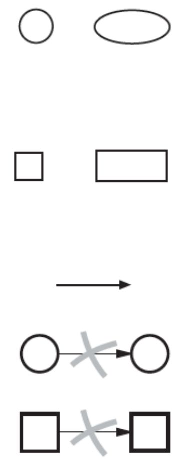
## Components of a net

A Petri net is a structure with two kinds of elements: places and transitions. They are connected by arcs.

A place is represented by a circle or ellipse. A place $p$ always models a passive component: $p$ can store, accumulate or show things.

A transition is represented by a square or rectangle. A transition $t$ always models an active component: $t$ can produce things, consume, transport or change them.

Places and transitions are connected to each other by directed arcs. Graphically, an arc is represented by an arrow. An arc models an abstract, sometimes only notional relation between components. Arcs run from places to transitions or vice versa.



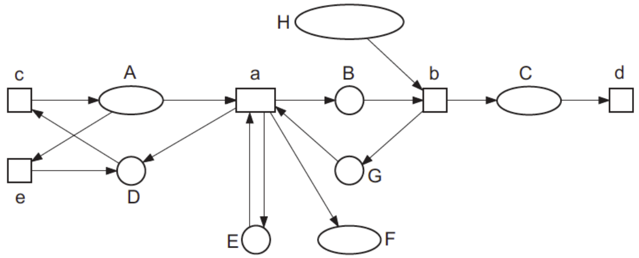
# Nets

## Net

A Petri net $N$ is a triple $(P, T, F)$ where:

- $P$ is the finite set of places
- $T$ is the finite set of transitions with $P \cap T = \varnothing$
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs[2]

Places and transitions are generically called nodes.




[2] $F$ is also called the flow relation.



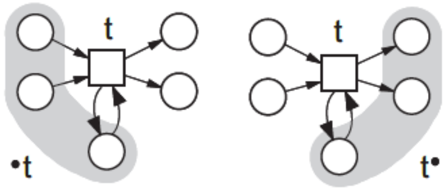


# The pre- and post-sets

## Pre- and post-sets

Let node $x \in P \cup T$.
The pre-set of $x$ is defined by: ${}^\bullet x = \{ y \mid (y, x) \in F \}$.
The post-set of $x$ is defined by: $x^\bullet = \{ y \mid (x, y) \in F \}$.

Two nodes $x, y \in N$ form a loop if $x \in {}^\bullet y$ and $y \in {}^\bullet x$.



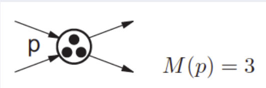





# Markings

## Marking

A marking $M$ of a net $N = (P, T, F)$ is a mapping $M : P \to \mathbb{N}$.
For net $N = (P, T, F)$ and marking $M_0$, the tuple $(P, T, F, M_0)$ is called an elementary system net. $M_0$ is the initial marking of $N$.




## Intuition

Note: a marking is a multiset. It defines a distribution of tokens across places. Tokens are depicted as black dots.





# Transition firing

## Enabling and occurrence of a transition

Let $(P, T, F, M)$ be an elementary system net. Marking $M$ enables a transition $t$ if $M(p) \geqslant 1$ for each place $p \in^\bullet t$.

Transition $t$ can occur in marking $M$ if $t$ is enabled at $M$. Its occurrence leads to marking $M'$, denoted $M \xrightarrow{t} M'$, defined for place $p \in P$ by:

$$M'(p) \;=\; M(p) - F(p, t) + F(t, p).$$

where we represent $F$ by its characteristic function.

## Intuition

Transition $t$ is enabled whenever every $p \in^\bullet t$ holds at least one token. On $t$'s occurrence, one token is removed from each place in ${}^\bullet t$, and one token is put in each place in $t^\bullet$:

$$M'(p) \;=\; \begin{cases} M(p) - 1 & \text{if } p \in {}^\bullet t \text{ and } p \notin t^\bullet \\ M(p) + 1 & \text{if } p \in t^\bullet \text{ and } p \notin^\bullet t \\ M(p) & \text{otherwise} \end{cases}$$

## Transition occurrence

**Enabling and occurrence of a transition**

A marking $M$ enables a transition $t$ if $M(p) \geqslant 1$ for each place $p \in {}^\bullet t$.

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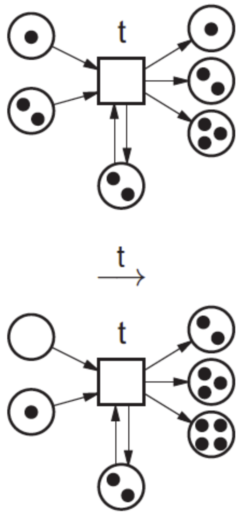
where we represent $F$ by its characteristic function.
$M \xrightarrow{t} M'$ is also called a step of the net $N$.








## Overview







## The interleaving semantics of Petri nets

An execution semantics

State: marking (distribution of tokens over the net)

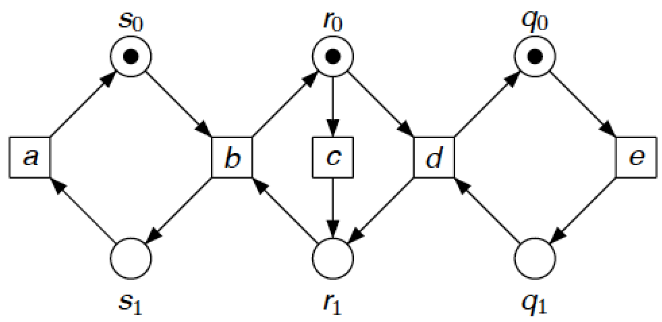
Transitions: $M \xrightarrow{t} M'$

Sequential runs: $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$





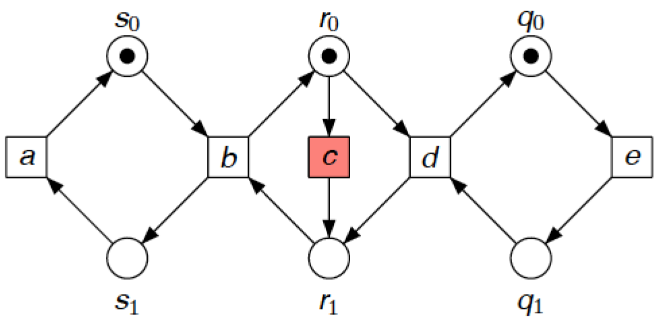
## The interleaving semantics of Petri nets




$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As the marking for $s_0$ is the complement of $s_1$, the marking for $s_0$ is omitted. The same applies to the places $r_0$ and $q_0$.

## The interleaving semantics of Petri nets




$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c}$$



Lecture 17: Interleaving Semantics of Petri Nets | The interleaving semantics of Petri nets

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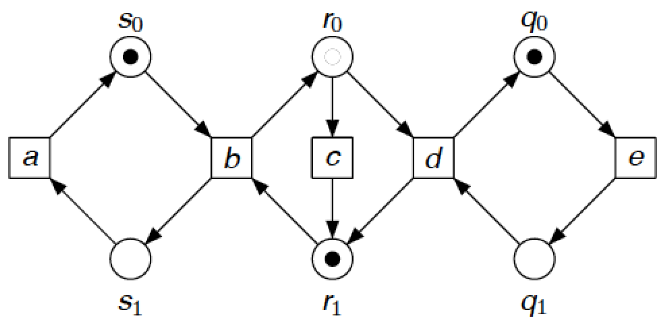


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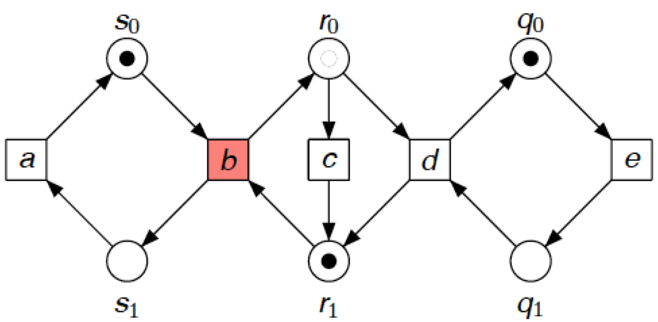


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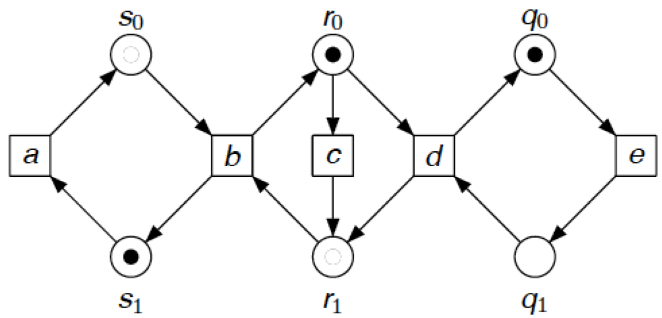


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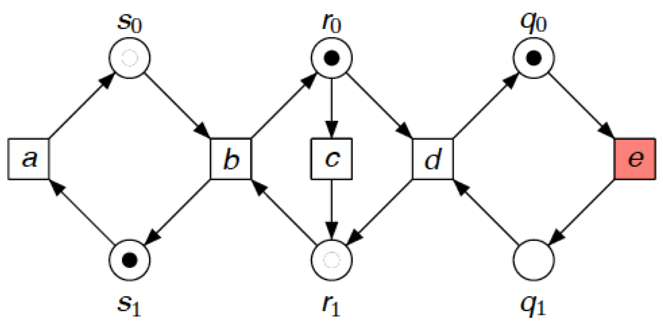


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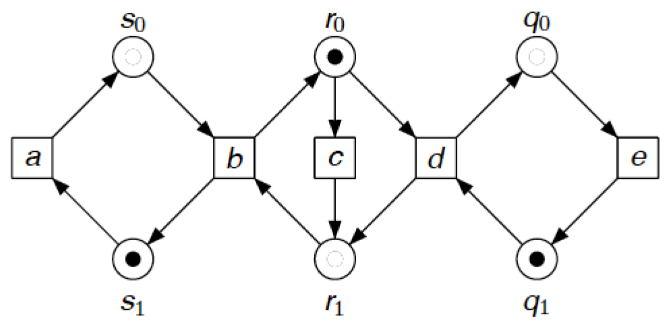


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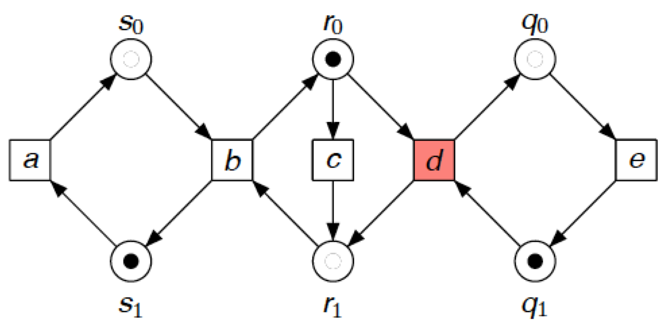


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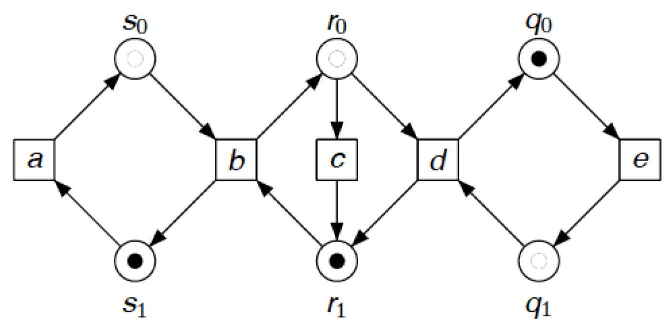


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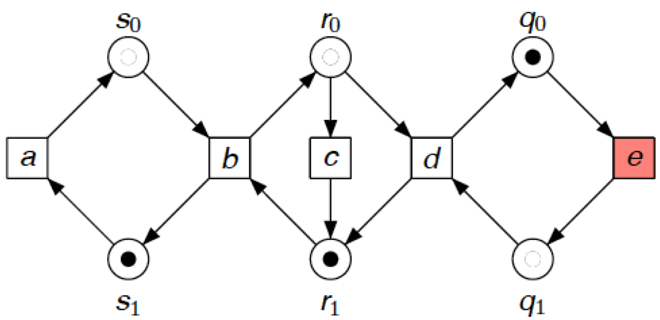


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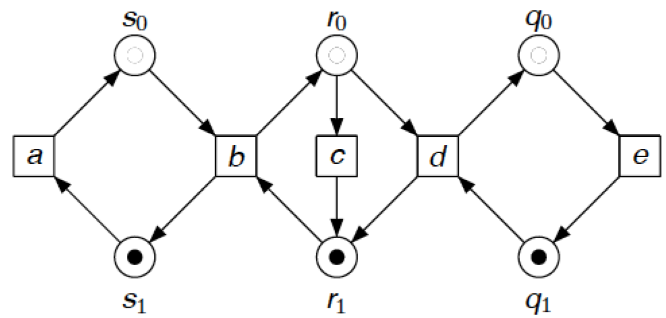


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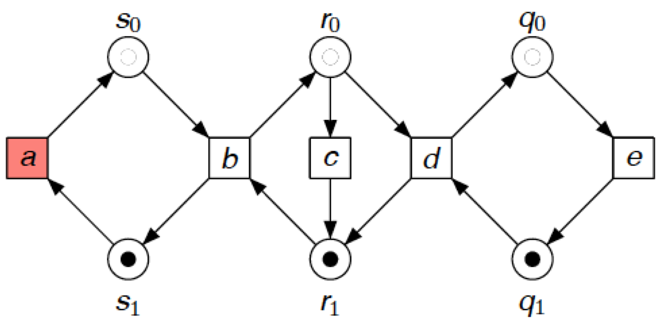


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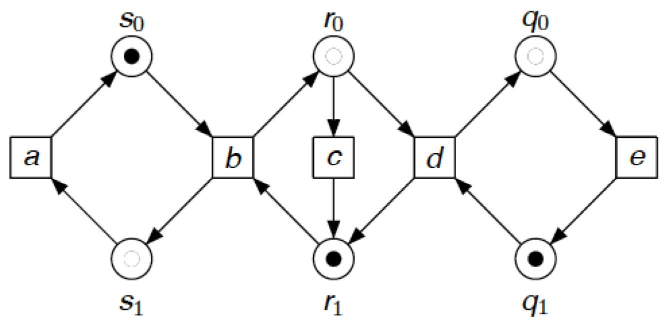
# The interleaving semantics of Petri nets

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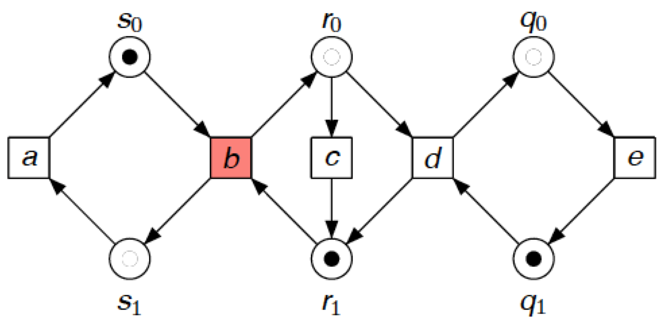
## The interleaving semantics of Petri nets

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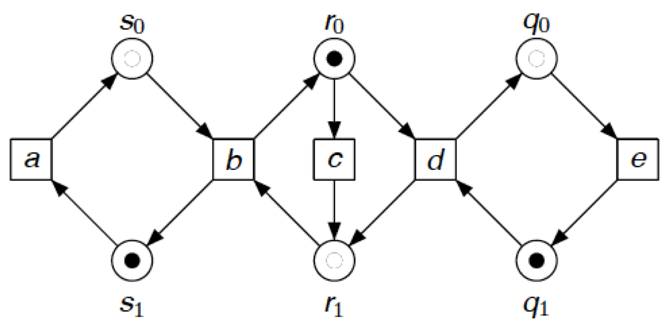
## The interleaving semantics of Petri nets

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## Reachable markings

**Step sequence**

A sequence of transitions $\sigma = t_1\, t_2 \ldots t_n$ is an step sequence if there exist markings $M_1$ through $M_n$ such that:

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M_n$$

Marking $M_n$ is reached by the occurrence of $\sigma$, denoted $M_0 \xrightarrow{\sigma} M_n$.

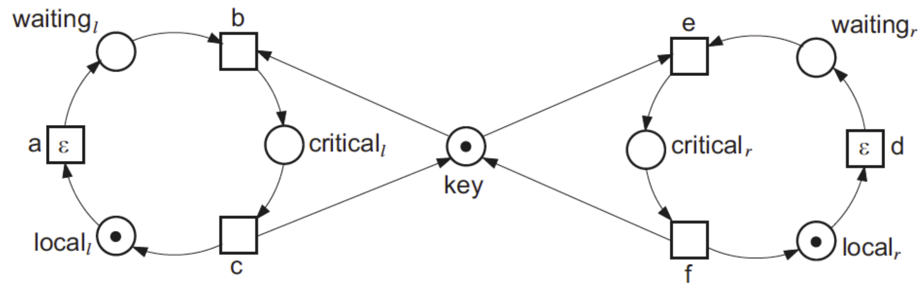
$M$ is a reachable marking if there exists a step sequence $\sigma$ with $M_0 \xrightarrow{\sigma} M$.





## Mutual exclusion

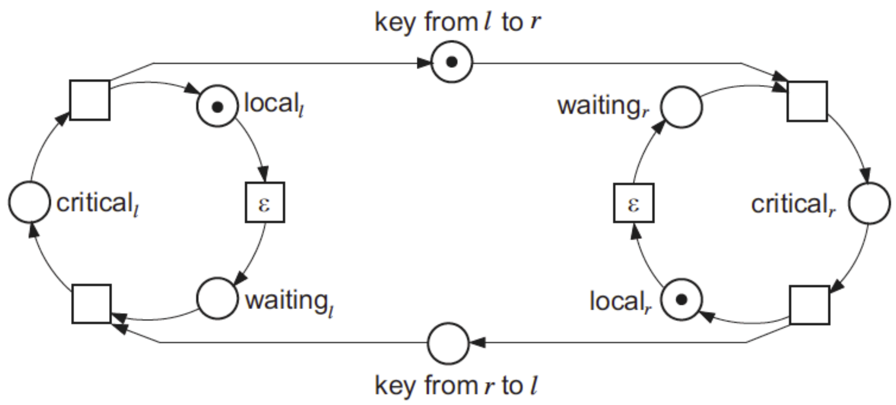
Two processes cycling through the states local, waiting and critical.




Between transitions b and e a conflict can arise infinitely often. No strategy has been modeled to solve this conflict.

## Mutual exclusion

A strategy where processes are acquired access in an alternating fashion:







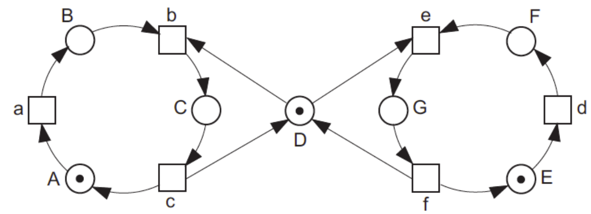

## One-bounded elementary system nets

### 1-bounded elementary net system

An elementary net system $N$ is called 1-bounded if for each reachable marking $M$ and place $p$ of $N$:

$$M(p) \leqslant 1.$$

Markings of 1-bounded elementary net systems can be described as a string of marked places, e.g., $ADE$. Two steps begin with this marking: $ADE \xrightarrow{a} BDE$ and $ADE \xrightarrow{d} ADF$.








## Overview







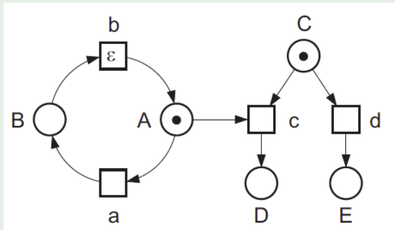
## Sequential runs

### Sequential run

Let $N$ be an elementary net system. A sequential run of $N$ is a sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots$$

of steps of $N$ starting with the initial marking $M_0$. A run can be finite or infinite. A finite run $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_1} \cdot \xrightarrow{t_n} M_n$ is complete if $M_n$ does not enable any transition.




A sample complete run is:

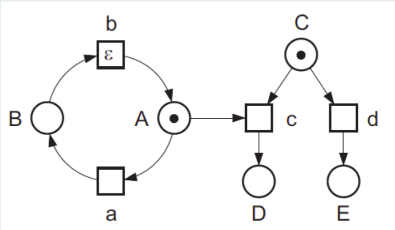
$$AC \xrightarrow{a} BC \xrightarrow{b} AC \xrightarrow{c} D$$

A sample incomplete run is:

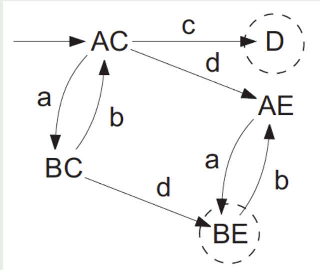
$$AC \xrightarrow{d} AE \xrightarrow{a} BE$$

## Marking graph

The marking graph of $N$ has as nodes the reachable markings of $N$ and as edges the reachable steps of $N$.




A sample elementary net system




Its marking graph





## Overview







## Summary

- A Petri net consists of places, transitions and arcs
- An elementary net is a Petri net plus a marking
- Firing a single transition in a marking is a step
- A sequential run is a sequence of steps starting in the initial marking
- A marking graph has as nodes the reachable markings of the net and as edges its reachable steps.
- The marking graph is the interleaving semantics of a net.