Concurrency Theory

Weak bisimulation

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http://moves.rwth-aachen.de/teaching/ws-1718/ct

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Overview

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Aims of this lecture

1. A notion of bisimulation that treats τ-actions as unobservable
2. How to treat divergences, i.e., loops of τ-actions?
3. A slight adaptation that yields a CCS congruence

Overview

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The processes P and Q are strongly bisimilar, denoted $P \sim Q$, iff there is a strong bisimulation $\mathcal{R}$ with $(P, Q) \in \mathcal{R}$. Thus,

$$\sim \;=\; \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \}.$$

Relation $\sim$ is called strong bisimilarity.

Properties of strong bisimilarity

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1. $\sim$ is an equivalence relation.
2. $P \sim Q \implies Tr(P) = Tr(Q)$.
3. $\sim$ is a CCS congruence.
4. $\sim$ is deadlock sensitive.
5. checking $\sim$ is decidable for finite-state processes and can be done in polynomial time.[1]
6. $\sim$ has a nice game characterization.

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Sequential two-place buffer

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Parallel two-place buffer

$$\begin{array}{rcl} pB & = & (oB[f] \,||\, oB[g]) \setminus \{ com \} \\ & & \text{with } f(in) = in \text{ and } f(out) = com \\ & & \ \text{and } g(in) = com \text{ and } g(out) = out \\ oB & = & in.\overline{out}.oB \end{array}$$

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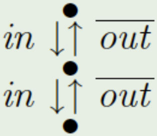
Sequential buffer $\not\sim$ parallel buffer



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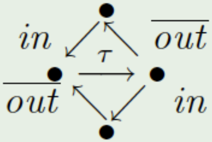

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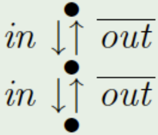

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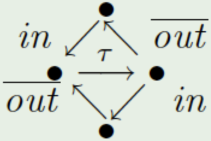

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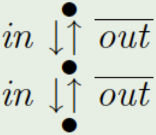

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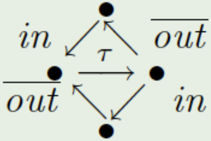

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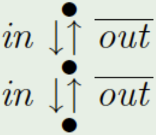

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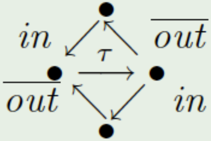

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Rationale: τ-actions are special as they are unobservable.

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- Strong bisimilarity treats τ-actions as any other action.
- Can we yield the nice properties of $\sim$ while "abstracting" from τ-actions?

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Weak bisimulation [Milner, 1989]

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$$\approx \; = \; \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a weak bisimulation} \}.$$

Relation $\approx$ is called an observational equivalence or weak bisimilarity.

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$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \mathcal{R} & & \\ Q & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \mathcal{R} & & {\color{red}\mathcal{R}} \\ Q & {\color{red}\overset{\alpha}{\Longrightarrow}} & {\color{red}Q'} \end{array}$$

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Remark

Each clause in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  implies ex. $Q' \in Prc$ with $Q \; (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* \; Q'$ and $(P', Q') \in \mathcal{R}$
- $P \xrightarrow{\tau} P'$
  implies ex. $Q' \in Prc$ such that $Q \; (\xrightarrow{\tau})^* \; Q'$ and $(P', Q') \in \mathcal{R}$
  (where $Q' = Q$ is admissible)

Examples

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A first example

Let $P = \tau.a.\text{nil}$ and $Q = a.\text{nil}$. Then $P \not\sim Q$. Claim: $P \approx Q$. Rewrite P as: $P = \tau.P_1$ with $P_1 = a.\text{nil}$. Let $\mathcal{R} = \{ (P, Q), (P_1, Q), (\text{nil}, \text{nil}) \}$. Check that $\mathcal{R}$ is a weak bisimulation. As $(P, Q) \in \mathcal{R}$, it follows $P \approx Q$.

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Buffers

Check that the parallel and sequential buffer are weakly bisimilar.

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2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \overset{\tau}{\Longrightarrow} P$ with $(P, P) \in \mathcal{R}$ (since $id_{Prc} \subseteq \mathcal{R}$).

$\square$

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Note that also $Div \approx \mathrm{nil}$ where $Div = \tau.Div$. Thus, a deadlock process is weakly bisimilar to a process that can only diverge. This is justified by the fact that "observations" can only be made by interacting with the process.

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Properties of weak bisimilarity

Properties of $\approx$

1. $P \sim Q$ implies $P \approx Q$.
2. $\approx$ is an equivalence relation (reflexive, symmetric, transitive).
3. $\approx$ is the largest weak bisimulation.
4. $\approx$ is (non-τ) deadlock sensitive.[3]
5. $\approx$ abstracts from τ-loops.

[3] Where w-deadlocks are considered on observable traces w.

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Proof.

1. Straightforward. 2.–4. Similar to the proofs for $\sim$. Left as as exercise.
5. Previous slide. □

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Weak bisimilarity versus trace equivalence

Weak bisimilarity versus trace equivalence

Observational trace language

The observational trace language of $P \in Prc$ is defined by:

$$ObsTr(P) \; = \; \{ \, \widehat{w} \in Act^* \mid \exists P' \in Prc.\, P \xrightarrow{w} P' \, \}$$

where $\widehat{w}$ is obtained from w by omitting all τ-actions.

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Theorem

$P \approx Q$ implies that P and Q are observational trace equivalent. The reverse does not hold.

Milner's τ-laws

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$$\begin{array}{rcl} \alpha.\tau.P & \approx & \alpha.P \\ P + \tau.P & \approx & \tau.P \\ \alpha.(P + \tau.Q) & \approx & \alpha.(P + \tau.Q) + \alpha.Q. \end{array}$$

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Proof.

Left as an exercise. Build appropriate weak bisimulation relations. □

Congruence

CCS congruence

Let $P, Q \in Prc$ be CCS processes. Assume $P \approx Q$. Then:

$$\begin{array}{rcl} \alpha.P & \approx & \alpha.Q \text{ for every action } \alpha \\ P||R & \approx & Q||R \text{ for every process } R \\ P\backslash L & \approx & Q\backslash L \text{ for every set } L \subseteq A \\ P[f] & \approx & Q[f] \text{ for every relabelling } f. \end{array}$$

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What about choice?

$\tau.a.\mathrm{nil} \approx a.\mathrm{nil}$ but $\tau.a.\mathrm{nil} + b.\mathrm{nil} \not\approx a.\mathrm{nil} + b.\mathrm{nil}$.

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Thus, weak bisimilarity is not a congruence for CCS.
This motivates a slight adaptation of $\approx$.

Overview

Observation congruence

Observation congruence [Milner, 1989]

$P, Q \in \mathit{Prc}$ are observationally congruent, denoted $P \approx^c Q$, if for every $\alpha \in \mathit{Act}$ (including $\alpha = \tau$):

1. if $P \xrightarrow{\alpha} P'$ then there is a sequence of transitions $Q \stackrel{\tau}{\Longrightarrow} Q_1 \xrightarrow{\alpha} Q_2 \stackrel{\tau}{\Longrightarrow} Q'$ such that $P' \approx Q'$

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$\approx^c$ differs from $\approx$ only in that $\approx^c$ requires τ-moves by P or Q to be mimicked by at least one τ-move in the other process.

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$\approx^c$ differs from $\approx$ only in that $\approx^c$ requires τ-moves by P or Q to be mimicked by at least one τ-move in the other process. This only applies to the first step; the successors just have to satisfy $P' \approx Q'$ (and not necessarily $P' \approx^c Q'$).

Examples

Example

1. Sequential and parallel two-place buffer:

$$
\begin{array}{c}
P_1 \\
in \downarrow\uparrow \overline{out} \\
P_2 \\
in \downarrow\uparrow \overline{out} \\
P_3
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & Q_1 & \\
in \swarrow & & \nwarrow \overline{out} \\
Q_2 & \xrightarrow{\tau} & Q_3 \\
\overline{out} \nwarrow & & \swarrow in \\
 & Q_4 &
\end{array}
$$

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\begin{tikzcd}
 & Q_1 \arrow[dl, "in"'] & \\
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3. $b.\tau.\mathrm{nil} \approx^c b.\mathrm{nil}$ (since $\tau.\mathrm{nil} \approx \mathrm{nil}$).

Properties of observation congruence

Theorem

For every $P, Q \in \mathit{Prc}$, it holds:

1. $P \sim Q$ implies $P \approx^c Q$, and $P \approx^c Q$ implies $P \approx Q$
2. $\approx^c$ is a CCS congruence
3. $P \approx^c Q$ if and only if $P + R \approx Q + R$ for every $R \in \mathit{Prc}$
4. $\approx^c$ is an equivalence relation
5. $P \approx Q$ if and only if ($P \approx^c Q$ or $P \approx^c \tau.Q$ or $\tau.P \approx^c Q$)

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Proof.

Omitted. □

Note: as $\approx$ implies trace equivalence and is (non-τ) deadlock-sensitive, $\approx^c$ implies trace equivalence and is (non-τ) deadlock-sensitive.

Overview

Weak bisimilarity as a game

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Game rules

In each round the current configuration (s, t) is changed as follows:

1. the attacker chooses one of the processes in the current configuration, say t, and makes an $\xrightarrow{\alpha}$-move for some $\alpha \in Act$ to t', say, and

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2. the defender must respond by making an $\stackrel{\alpha}{\Longrightarrow}$-move in the other process s of the current configuration under the same action α, yielding $s \stackrel{\alpha}{\Longrightarrow} s'$.

The new pair of processes (s', t') becomes the current configuration. The game continues with another round.

Game results

1. If one player cannot move, the other player wins.
2. If the game can be played *ad infinitum*, the defender wins.

Game characterization of weak bisimilarity

Theorem **[Stirling, 1995], [Thomas, 1993]**

1. $s \approx t$ iff the defender has a universal winning strategy from configuration (s, t).
2. $s \not\approx t$ iff the attacker has a universal winning strategy from configuration (s, t).

(By means of a universal winning strategy, a player can always win, regardless of how the other player selects her moves.)

Proof.

Similar as for strong bisimilarity. Left as an exercise. □

Deciding weak bisimilarity

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Checking whether $P \approx Q$ (or $P \approx^c Q$) over finite-state processes can be reduced to checking strong bisimilarity $\sim$, using a technique called saturation.

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2. Constructing a new pair of finite-state processes whose original transitions are replaced by weak transitions.

The question whether $P \approx Q$ now boils down to checking $\sim$ on the saturated processes. (Details are outside the scope of this lecture.)

As computing $\Longrightarrow$ and $\sim$ can be done in polynomial time, $P \approx Q$ can be checked in polynomial time.

Overview

Summary

1. Weak bisimilarity is based on mutual mimicking processes
2. But: τ-actions do not need to be mimicked, as they are internal
3. Weak bisimilarity is not a congruence for choice $(+)$
4. Observation congruence remedies this by forcing initial τ-actions to be mimicked
5. Divergence is weakly bisimilar to a deadlock process
6. Checking (non-)weak bisimilarity can be done using a two-player game
7. Weak bisimilarity can be determined in polynomial-time