Concurrency Theory

Weak bisimulation

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Overview

Overview

Summary so far

- Strong bisimulation is based on mutual mimicking of processes
- Strong bisimilarity ($\sim$) is a congruence, is deadlock sensitive
- Implies trace equivalence, and can be computed in polynomial time

But $\sim$ does not distinguish between internal (τ-) actions and observable actions.

Aims of this lecture

1. A notion of bisimulation that treats τ-actions as unobservable
2. How to treat divergences, i.e., loops of τ-actions?
3. A slight adaptation that yields a CCS congruence

Overview

Strong bisimulation

Strong bisimulation **[Park, 1981, Milner, 1989]**

A binary relation $\mathcal{R} \subseteq Prc \times Prc$ is a strong bisimulation whenever for every $(P, Q) \in \mathcal{R}$, and $\alpha \in Act$:

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \xrightarrow{\alpha} P'$ and $(P', Q') \in \mathcal{R}$.

Strong bisimilarity

The processes P and Q are strongly bisimilar, denoted $P \sim Q$, iff there is a strong bisimulation $\mathcal{R}$ with $(P, Q) \in \mathcal{R}$. Thus,

$$\sim \; = \; \bigcup \{\, \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \,\}.$$

Relation $\sim$ is called strong bisimilarity.

Properties of strong bisimilarity

Properties of strong bisimilarity

1. $\sim$ is an equivalence relation.
2. $P \sim Q \implies Tr(P) = Tr(Q)$.
3. $\sim$ is a CCS congruence.
4. $\sim$ is deadlock sensitive.
5. checking $\sim$ is decidable for finite-state processes and can be done in polynomial time.[1]
6. $\sim$ has a nice game characterization.

Question: is there a need to consider another behavioural equivalence?
Yes.

[1] In fact, computing $\sim$ is P-complete. It is thus one of the "hardest problems" admitting a polynomial-time algorithm.

Inadequacy of strong bisimilarity

Sequential two-place buffer

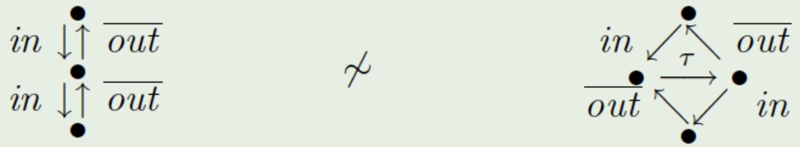
$$\begin{aligned} sB_0 &= in.sB_1 \\ sB_1 &= in.sB_2 + \overline{out}.sB_0 \\ sB_2 &= \overline{out}.sB_1. \end{aligned}$$

Parallel two-place buffer

$$\begin{aligned} pB &= (oB[f] \,||\, oB[g]) \setminus \{\, com \,\} \\ &\quad \text{with } f(in) = in \text{ and } f(out) = com \\ &\quad \text{ and } g(in) = com \text{ and } g(out) = out \\ oB &= in.\overline{out}.oB \end{aligned}$$

Inadequacy of strong bisimilarity

Sequential buffer $\not\sim$ parallel buffer



Problem: the sequential buffer cannot simulate the (invisible) τ-action.

Thus, the requirement in $\sim$ to exactly match all actions is often too strong.

This suggests to weaken this and not insist on exact matching of τ-actions.

Rationale: τ-actions are special as they are unobservable.

The rationales for abstracting from τ

- The τ-actions are internal and thus unobservable.
- This is natural in parallel communication yielding τ:
  synchronization in CCS is binary and as observation
  means communication with the process,
  the result of any communication is unobservable
- Strong bisimilarity treats τ-actions as any other action.
- Can we yield the nice properties of $\sim$ while "abstracting" from τ-actions?

Overview

Weak transition relation

Weak transition relation

$$\overset{\alpha}{\Longrightarrow} \;=\; \begin{cases} (\overset{\tau}{\longrightarrow})^* \circ \overset{\alpha}{\longrightarrow} \circ (\overset{\tau}{\longrightarrow})^* & \text{if } \alpha \neq \tau \\ (\overset{\tau}{\longrightarrow})^* & \text{if } \alpha = \tau. \end{cases}$$

where $(\overset{\tau}{\longrightarrow})^*$ is the reflexive and transitive closure of the relation $\overset{\tau}{\longrightarrow}$.

Informal meaning

1. If $\alpha \neq \tau$, then $s \overset{\alpha}{\Longrightarrow} t$ means that
   from s we can get to t by doing zero or more τ actions, followed by the action α, followed by zero or more τ actions.
2. If $\alpha = \tau$, then $s \overset{\alpha}{\Longrightarrow} t$ means that
   from s we can reach t by doing zero or more τ actions.

Weak bisimulation

Weak bisimulation **[Milner, 1989]**

A binary relation $\mathcal{R} \subseteq Prc \times Prc$ is a weak bisimulation whenever for every $(P, Q) \in \mathcal{R}$, and $\alpha \in Act$ (including $\alpha = \tau$):

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \overset{\alpha}{\Longrightarrow} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \overset{\alpha}{\Longrightarrow} P'$ and $(P', Q') \in \mathcal{R}$.

Weak bisimilarity

The processes P and Q are weakly bisimilar, denoted $P \approx Q$, iff there is a weak bisimulation $\mathcal{R}$ with $(P, Q) \in \mathcal{R}$. Thus,

$$\approx \; = \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a weak bisimulation} \}.$$

Relation $\approx$ is called an observational equivalence or weak bisimilarity.

Weak bisimulation

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \mathcal{R} & & \\ Q & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \mathcal{R} & & \mathcal{R} \\ Q & \overset{\alpha}{\Longrightarrow} & Q' \end{array}$$

and

$$\begin{array}{ccc} P & & \\ \mathcal{R} & & \\ Q & \xrightarrow{\alpha} & Q' \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} P & \overset{\alpha}{\Longrightarrow} & P' \\ \mathcal{R} & & \mathcal{R} \\ Q & \xrightarrow{\alpha} & Q' \end{array}$$

Explanation

Weak bisimulation **[Milner, 1989]**

A binary relation $\mathcal{R} \subseteq Prc \times Prc$ is a weak bisimulation whenever for every $(P, Q) \in \mathcal{R}$, and $\alpha \in Act$ (including $\alpha = \tau$):

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \overset{\alpha}{\Longrightarrow} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \overset{\alpha}{\Longrightarrow} P'$ and $(P', Q') \in \mathcal{R}$.

Remark

Each clause in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  implies ex. $Q' \in Prc$ with $Q \, (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* \, Q'$ and $(P', Q') \in \mathcal{R}$
- $P \xrightarrow{\tau} P'$
  implies ex. $Q' \in Prc$ such that $Q \, (\xrightarrow{\tau})^* \, Q'$ and $(P', Q') \in \mathcal{R}$
  (where $Q' = Q$ is admissible)

Examples

Weak bisimulation **[Milner, 1989]**

A binary relation $\mathcal{R} \subseteq Prc \times Prc$ is a weak bisimulation whenever for every $(P, Q) \in \mathcal{R}$, and $\alpha \in Act$ (including $\alpha = \tau$):

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \overset{\alpha}{\Longrightarrow} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \overset{\alpha}{\Longrightarrow} P'$ and $(P', Q') \in \mathcal{R}$.

A first example

Let $P = \tau.a.\text{nil}$ and $Q = a.\text{nil}$. Then $P \not\sim Q$. Claim: $P \approx Q$. Rewrite P as: $P = \tau.P_1$ with $P_1 = a.\text{nil}$. Let $\mathcal{R} = \{ (P, Q), (P_1, Q), (\text{nil}, \text{nil}) \}$. Check that $\mathcal{R}$ is a weak bisimulation. As $(P, Q) \in \mathcal{R}$, it follows $P \approx Q$.

Buffers

Check that the parallel and sequential buffer are weakly bisimilar.

Overview

Divergence

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\mathcal{R} \;=\; \{\,(P, \tau.P)\,\} \cup id_{Prc}$$

is a weak bisimulation:

1. every transition of P, $P \xrightarrow{\alpha} P'$
   can be simulated by $\tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   i.e., equivalently $\tau.P \overset{\alpha}{\Longrightarrow} P'$ with $(P', P') \in \mathcal{R}$ (since $id_{Prc} \subseteq \mathcal{R}$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \overset{\tau}{\Longrightarrow} P$ with $(P, P) \in \mathcal{R}$ (since $id_{Prc} \subseteq \mathcal{R}$).

□

Divergence

A polling process **[Koomen, 1982]**

Let:

$$A? \;=\; a.\text{nil} + \tau.B?$$
$$B? \;=\; b.\text{nil} + \tau.A?$$

Claim: $A? \approx B? \approx a.\text{nil} + b.\text{nil}$. (Check this!)

But note that $A? \xrightarrow{\tau} B? \xrightarrow{\tau} A?$ is a τ-loop, whereas $a.\text{nil} + b.\text{nil}$ does not have a loop, not even a τ-loop.

Thus, $\approx$ assumes that if a process can escape from a τ-loop, it eventually will do so.[2]

Note that also $Div \approx \text{nil}$ where $Div = \tau.Div$. Thus, a deadlock process is weakly bisimilar to a process that can only diverge. This is justified by the fact that "observations" can only be made by interacting with the process.

[2] This is called fair abstraction from divergence. Divergence is a τ-loop.

Properties of weak bisimilarity

Properties of $\approx$

1. $P \sim Q$ implies $P \approx Q$.
2. $\approx$ is an equivalence relation (reflexive, symmetric, transitive).
3. $\approx$ is the largest weak bisimulation.
4. $\approx$ is (non-τ) deadlock sensitive.[3]
5. $\approx$ abstracts from τ-loops.

Proof.

1. Straightforward. 2.–4. Similar to the proofs for $\sim$. Left as as exercise.
5. Previous slide. □

[3] Where w-deadlocks are considered on observable traces w.

Weak bisimilarity versus trace equivalence

Observational trace language

The observational trace language of $P \in Prc$ is defined by:

$$ObsTr(P) = \{ \widehat{w} \in Act^* \mid \exists P' \in Prc.\, P \xrightarrow{w} P' \}$$

where $\widehat{w}$ is obtained from w by omitting all τ-actions.

Trace equivalence

$P, Q \in Prc$ are observational trace equivalent if $ObsTr(P) = ObsTr(Q)$.

Theorem

$P \approx Q$ implies that P and Q are observational trace equivalent. The reverse does not hold.

Milner's τ-laws

Milner's τ-laws

$$\begin{aligned} \alpha.\tau.P &\approx \alpha.P \\ P + \tau.P &\approx \tau.P \\ \alpha.(P + \tau.Q) &\approx \alpha.(P + \tau.Q) + \alpha.Q. \end{aligned}$$

Proof.

Left as an exercise. Build appropriate weak bisimulation relations. □

Congruence

CCS congruence

Let $P, Q \in Prc$ be CCS processes. Assume $P \approx Q$. Then:

$$\begin{aligned} \alpha.P &\approx \alpha.Q \text{ for every action } \alpha \\ P||R &\approx Q||R \text{ for every process } R \\ P\backslash L &\approx Q\backslash L \text{ for every set } L \subseteq A \\ P[f] &\approx Q[f] \text{ for every relabelling } f. \end{aligned}$$

What about choice?

$\tau.a.\text{nil} \approx a.\text{nil}$ but $\tau.a.\text{nil} + b.\text{nil} \not\approx a.\text{nil} + b.\text{nil}$.

Thus, weak bisimilarity is not a congruence for CCS.
This motivates a slight adaptation of $\approx$.

Overview

Observation congruence

Observation congruence [Milner, 1989]

$P, Q \in Prc$ are observationally congruent, denoted $P \approx^c Q$, if for every $\alpha \in Act$ (including $\alpha = \tau$):

1. if $P \xrightarrow{\alpha} P'$ then there is a sequence of transitions $Q \stackrel{\tau}{\Longrightarrow} Q_1 \xrightarrow{\alpha} Q_2 \stackrel{\tau}{\Longrightarrow} Q'$ such that $P' \approx Q'$
2. if $Q \xrightarrow{\alpha} Q'$ then there is a sequence of transitions $P \stackrel{\tau}{\Longrightarrow} P_1 \xrightarrow{\alpha} P_2 \stackrel{\tau}{\Longrightarrow} P'$ such that $P' \approx Q'$.

Remark

$\approx^c$ differs from $\approx$ only in that $\approx^c$ requires τ-moves by P or Q to be mimicked by at least one τ-move in the other process. This only applies to the first step; the successors just have to satisfy $P' \approx Q'$ (and not necessarily $P' \approx^c Q'$).

Examples

Example

1. Sequential and parallel two-place buffer:

$$\begin{array}{ccc} & P_1 & \\ in & \downarrow\uparrow & \overline{out} \\ & P_2 & \\ in & \downarrow\uparrow & \overline{out} \\ & P_3 & \end{array} \qquad \begin{array}{ccc} & Q_1 & \\ in \swarrow & & \nwarrow \overline{out} \\ Q_2 & \xrightarrow{\tau} & Q_3 \\ \overline{out} \nwarrow & & \swarrow in \\ & Q_4 & \end{array}$$

   $P_1 \approx^c Q_1$ since $P_1 \approx Q_1$ and neither P_1 nor Q_1 has initial τ-steps.
2. $\tau.b.\mathrm{nil} \not\approx^c b.\mathrm{nil}$ (since $\tau.b.\mathrm{nil} \xrightarrow{\tau}$ but $b.\mathrm{nil} \stackrel{\tau}{\not\longrightarrow}$) thus the counterexample to congruence of $\approx$ for $+$ does not apply.
3. $b.\tau.\mathrm{nil} \approx^c b.\mathrm{nil}$ (since $\tau.\mathrm{nil} \approx \mathrm{nil}$).

Properties of observation congruence

Theorem

For every $P, Q \in Prc$, it holds:

1. $P \sim Q$ implies $P \approx^c Q$, and $P \approx^c Q$ implies $P \approx Q$
2. $\approx^c$ is a CCS congruence
3. $P \approx^c Q$ if and only if $P + R \approx Q + R$ for every $R \in Prc$
4. $\approx^c$ is an equivalence relation
5. $P \approx Q$ if and only if ($P \approx^c Q$ or $P \approx^c \tau.Q$ or $\tau.P \approx^c Q$)

Proof.

Omitted. □

Note: as $\approx$ implies trace equivalence and is (non-τ) deadlock-sensitive, $\approx^c$ implies trace equivalence and is (non-τ) deadlock-sensitive.

Overview

Weak bisimilarity as a game

Game rules

In each round the current configuration (s, t) is changed as follows:

1. the attacker chooses one of the processes in the current configuration, say t, and makes an $\xrightarrow{\alpha}$-move for some $\alpha \in Act$ to t', say, and
2. the defender must respond by making an $\stackrel{\alpha}{\Longrightarrow}$-move in the other process s of the current configuration under the same action α, yielding $s \stackrel{\alpha}{\Longrightarrow} s'$.

The new pair of processes (s', t') becomes the current configuration. The game continues with another round.

Game results

1. If one player cannot move, the other player wins.
2. If the game can be played *ad infinitum*, the defender wins.

Game characterization of weak bisimilarity

Theorem [Stirling, 1995], [Thomas, 1993]

1. $s \approx t$ iff the defender has a universal winning strategy from configuration (s, t).
2. $s \not\approx t$ iff the attacker has a universal winning strategy from configuration (s, t).

(By means of a universal winning strategy, a player can always win, regardless of how the other player selects her moves.)

Proof.

Similar as for strong bisimilarity. Left as an exercise. □

Deciding weak bisimilarity

Checking whether $P \approx Q$ (or $P \approx^c Q$) over finite-state processes can be reduced to checking strong bisimilarity $\sim$, using a technique called saturation.

Intuitively, saturation amounts to:

1. First pre-computing the weak transition relation $\Longrightarrow$, and then
2. Constructing a new pair of finite-state processes whose original transitions are replaced by weak transitions.

The question whether $P \approx Q$ now boils down to checking $\sim$ on the saturated processes. (Details are outside the scope of this lecture.)

As computing $\Longrightarrow$ and $\sim$ can be done in polynomial time, $P \approx Q$ can be checked in polynomial time.

Overview

Summary

1. Weak bisimilarity is based on mutual mimicking processes
2. But: τ-actions do not need to be mimicked, as they are internal
3. Weak bisimilarity is not a congruence for choice $(+)$
4. Observation congruence remedies this by forcing initial τ-actions to be mimicked
5. Divergence is weakly bisimilar to a deadlock process
6. Checking (non-)weak bisimilarity can be done using a two-player game
7. Weak bisimilarity can be determined in polynomial-time