Concurrency Theory

Lecture 13: Strong Bisimulation — Games and Fixed Points

Joost-Pieter Katoen and Thomas Noll

Lehrstuhl für Informatik 2
Software Modeling and Verification Group

http://moves.rwth-aachen.de/teaching/ws-1718/ct

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Aims of this lecture

1. Using games to show non-bisimilarity of two processes
2. Strong simulation: one-way bisimulation
3. Using fixed points to compute $\sim$

Strong bisimulation

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$$\sim \;=\; \bigcup \{\, \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \,\}.$$

Relation $\sim$ is called strong bisimilarity.

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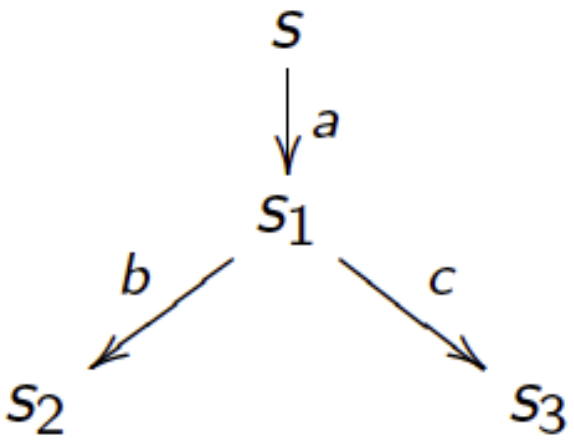

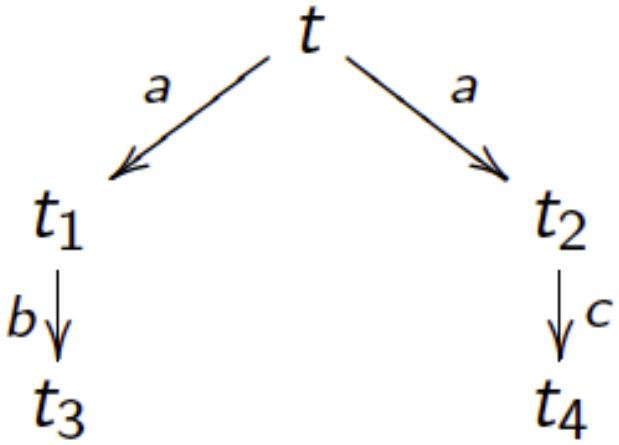

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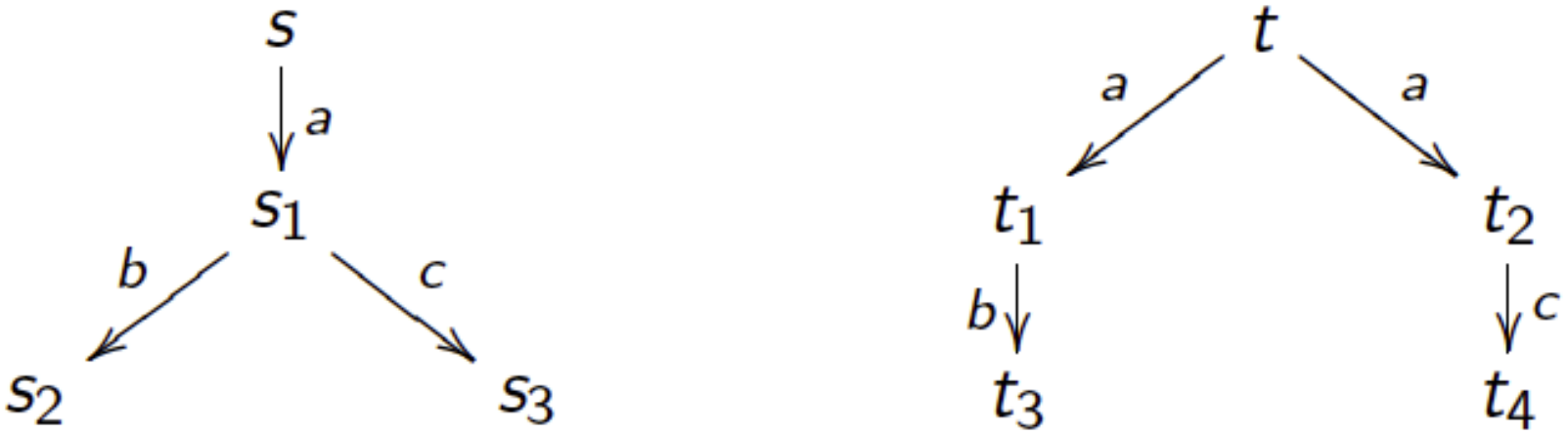

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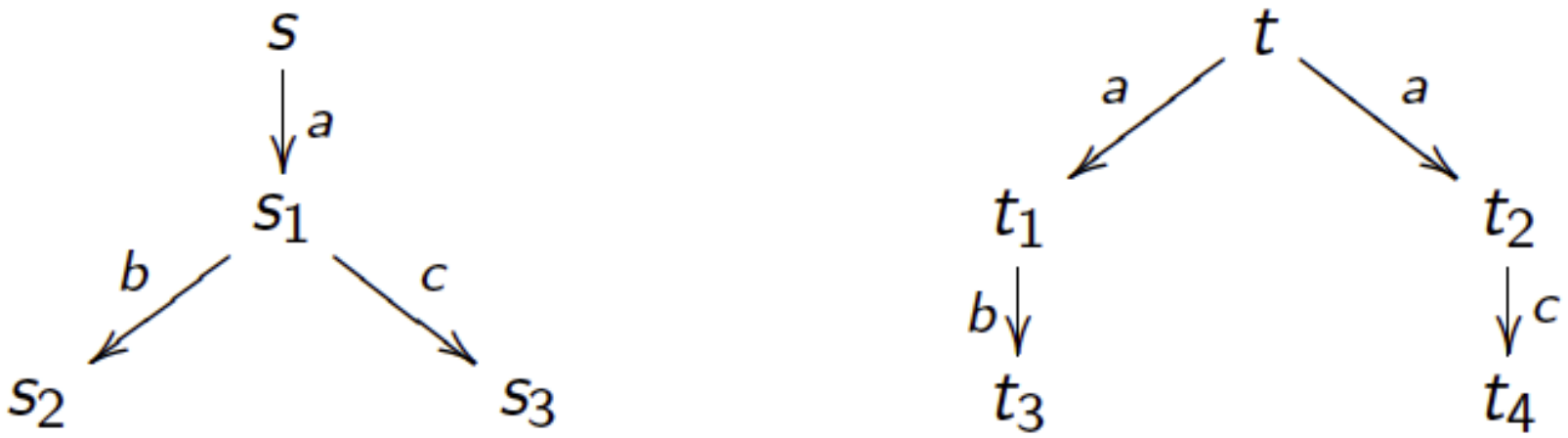

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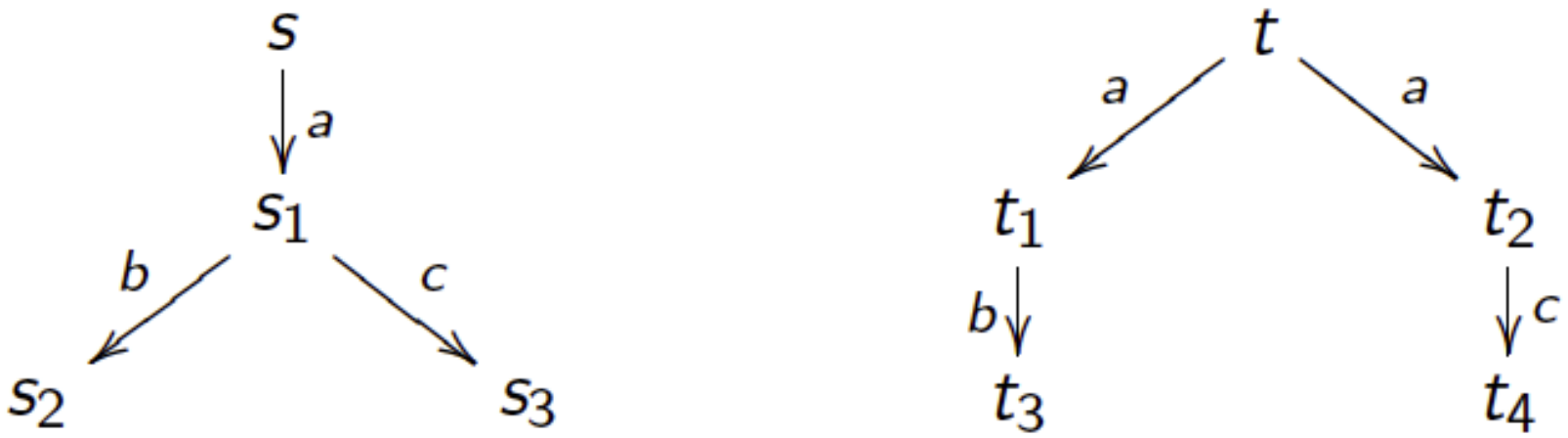

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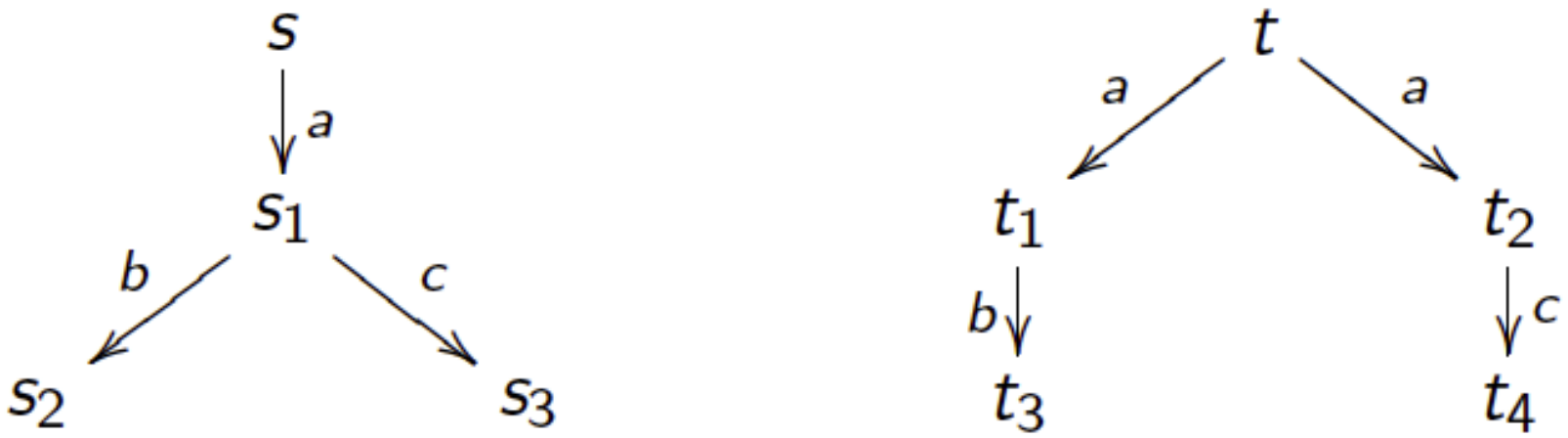

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- Make certain observations which will enable to disqualify many bisimulation candidates in one step.
- Use game characterization of strong bisimilarity.

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Intuition

The defender wants to show that $s \sim t$ while the attacker aims to show the opposite.

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Game results

1. If one player cannot move, the other player wins.
2. If the game can be played *ad infinitum*, the defender wins.

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Game characterization of bisimulation

Theorem **[Stirling, 1995], [Thomas, 1993]**

1. $s \sim t$ iff the defender has a universal winning strategy from configuration (s, t).
2. $s \not\sim t$ iff the attacker has a universal winning strategy from configuration (s, t).

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A bisimulation game can be used to prove bisimilarity as well as non-bisimilarity.[1] It often provides elegant arguments for $s \not\sim t$.

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Example

A first example

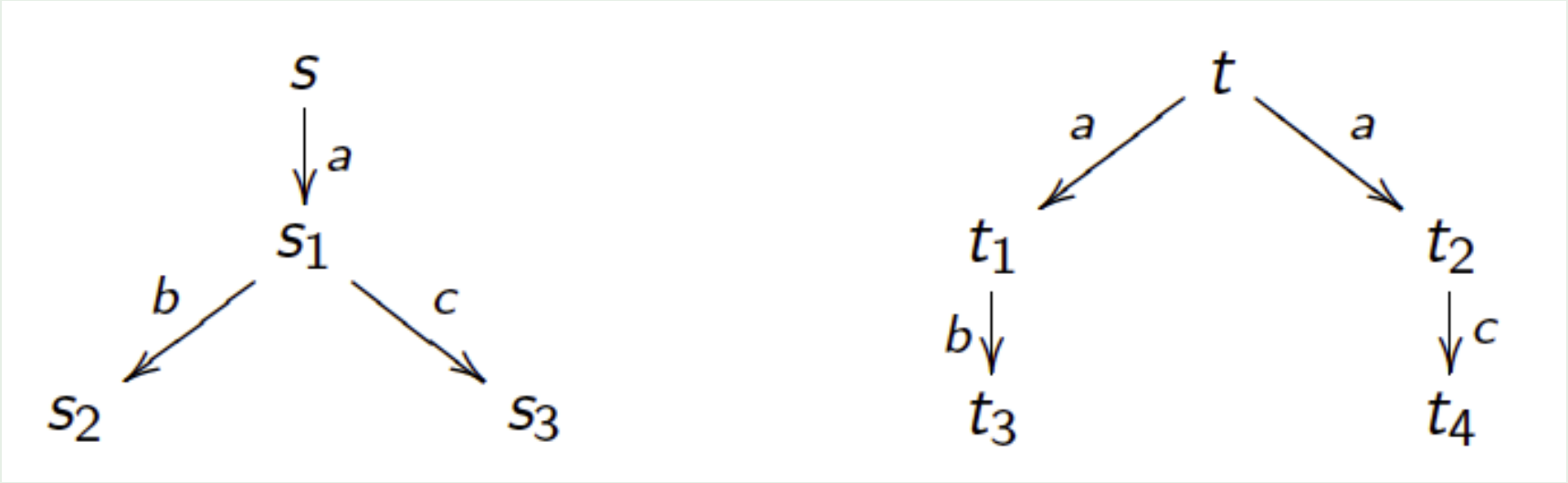
Use the game characterization to show $P \sim Q$ where:

$$\begin{aligned} P &= a.P_1 + a.P_2 \\ P_1 &= b.P_2 \\ P_2 &= b.P_2 \\ \\ Q &= a.Q_1 \\ Q_1 &= b.Q_1. \end{aligned}$$

Example

Another example

Use the game characterization to show that $s \not\sim t$ where:



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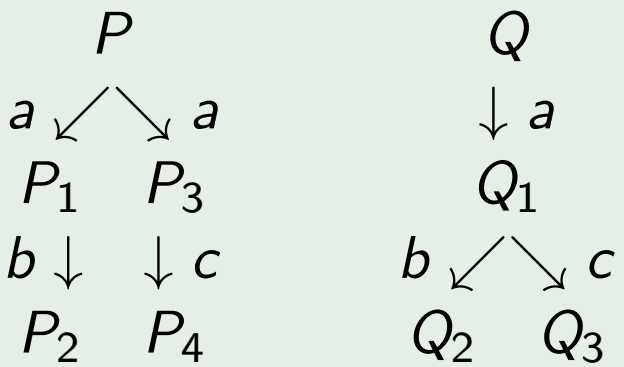

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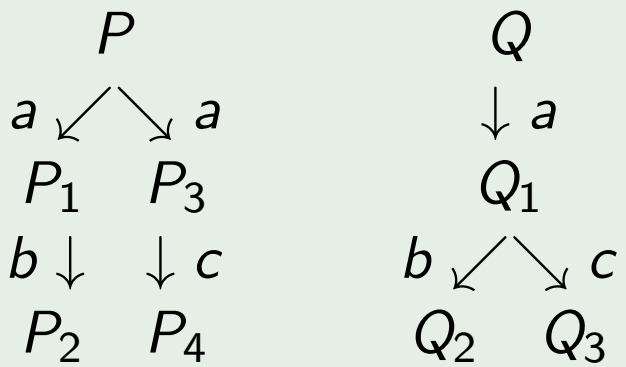

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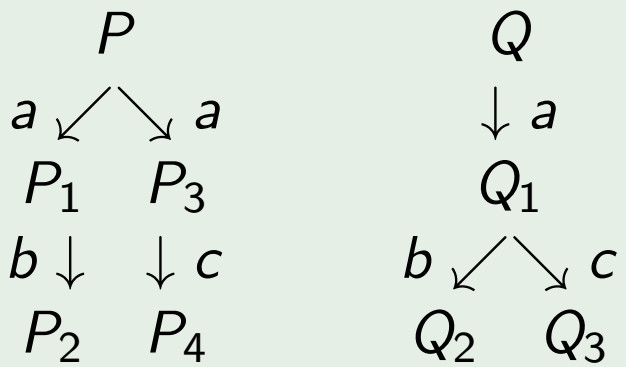

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This yields that:

$$a.b.\text{nil} + a.c.\text{nil} \sqsubseteq a.(b.\text{nil} + c.\text{nil})$$
$$a.(b.\text{nil} + c.\text{nil}) \not\sqsubseteq a.b.\text{nil} + a.c.\text{nil}.$$

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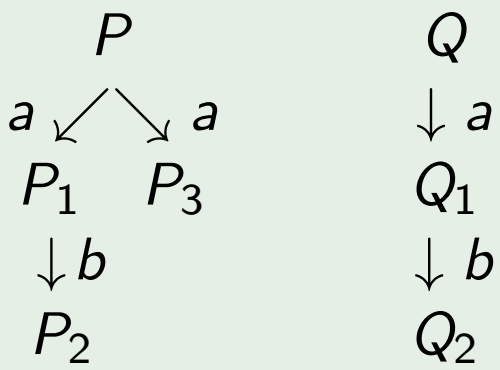

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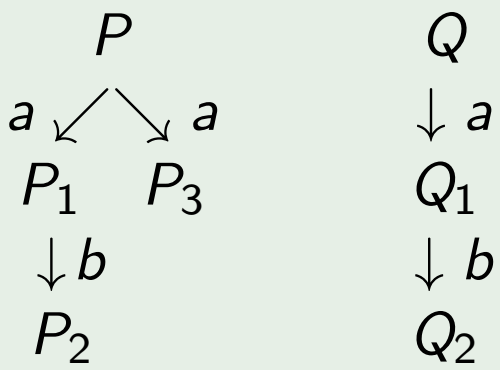

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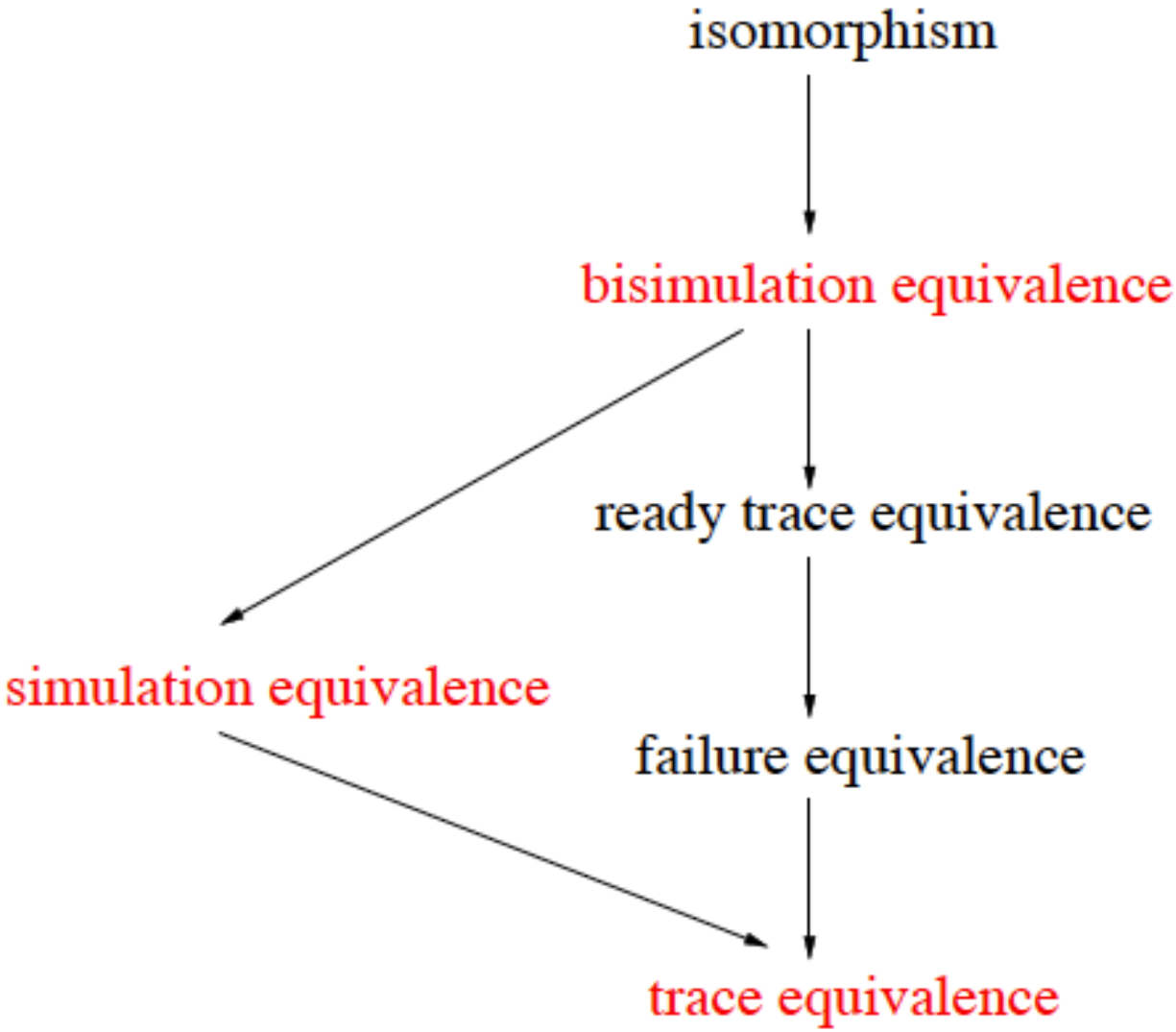
In addition to the requirement for $\sqsubseteq$, ready simulation requires that if Q does not deadlock on α, then P neither does.

Overview of some behavioural equivalences



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Note that $(2^{Prc \times Prc}, \subseteq)$ is a complete lattice[2] with $\bigcup$ and $\bigcap$ as least upper bound and greatest lower bound.

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Strong bisimilarity

Recall: $\sim$ implies trace equivalence, and checking trace equivalence is PSPACE-complete.

What about checking $\sim$ between two processes?

Strong bisimilarity

$$\sim = \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \}.$$

Note that $(2^{Prc \times Prc}, \subseteq)$ is a complete lattice[2] with $\bigcup$ and $\bigcap$ as least upper bound and greatest lower bound. We will show that $\sim$ can be characterized as a fixed point of a monotonic function on this lattice.

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Fixed point characterization of $\sim$

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Intuition

$\mathcal{F}(\mathcal{R})$ contains all pairs of processes from which, in one round of the bisimulation game, the defender can ensure that the players reach a current configuration that is contained in $\mathcal{R}$.

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Theorem

For finite-state process P with state space S, $\sim$ can be computed by:

$$\begin{array}{rcl} \sim & = & \bigcap_{i=0}^{\infty} \sim_i \quad \text{where } \sim_i \text{ is defined by} \\ \sim_0 & = & S \times S \\ \sim_{i+1} & = & \mathcal{F}(\sim_i). \end{array}$$

Proof.

Using the facts that $\mathcal{F}$ is monotonic on $(2^{\mathit{Prc}\times\mathit{Prc}}, \subseteq)$ and Tarski's fixed point theorem. □

Naive algorithm for checking $\sim$

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Naive algorithm for checking $\sim$

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Complexity **[Balcázar et al., 1992]**

Deciding strong bisimilarity between finite LTSs is P-complete.[3]

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Overview

Summary

1. Checking (non-)bisimilarity can be done using a two-player game
2. Strong simulation is a one-way strong bisimulation
3. Strong simulation equivalence is strictly coarser than $\sim$
4. Ready simulation takes deadlocks into account
5. Strong bisimilarity can be characterised as a fixed point
6. This yields a polynomial-time procedure for determining $\sim$.