



Concurrency Theory

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Lecture 10: Trace Equivalence

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<http://moves.rwth-aachen.de/teaching/ws-1718/ct/>

Introduction

- When using process algebras like CCS, an important approach is to model both the **specification and implementation** as CCS processes, say *Spec* and *Impl*.
- This gives rise to the natural question: when are two CCS processes **behaving the same**?
- As there are many different interpretations of “behaving the same”, **different behavioural equivalences** have emerged.

Introduction

Behavioural Equivalence

Implementation

$$CM = coin.\overline{coffee}.CM$$

$$CS = \overline{pub}.\overline{coin}.coffee.CS$$

$$Uni = (CM \parallel CS) \setminus \{coin, coffee\}$$

Specification

$$Spec = \overline{pub}.Spec$$

Question

Are the specification *Spec* and implementation *Uni* behaviourally equivalent:

$$Spec \stackrel{?}{\equiv} Uni$$

Equivalence Relations

Some reasonable required properties

- **Reflexivity:** $P \equiv P$ for every process P
- **Symmetry:** $P \equiv Q$ if and only if $Q \equiv P$
- **Transitivity:** $Spec_0 \equiv \dots \equiv Spec_n \equiv Impl$ implies that $Spec_0 \equiv Impl$

Definition 10.1 (Equivalence)

A binary relation $\equiv \subseteq S \times S$ over a set S is an **equivalence** if

- it is reflexive: $s \equiv s$ for every $s \in S$,
- it is symmetric: $s \equiv t$ implies $t \equiv s$ for every $s, t \in S$,
- it is transitive: $s \equiv t$ and $t \equiv u$ implies $s \equiv u$ for every $s, t, u \in S$.

Isomorphism: An Example Behavioural Equivalence

Definition 10.2 (LTS isomorphism)

Two LTSs $T_1 = (S_1, Act_1, \longrightarrow_1)$ and $T_2 = (S_2, Act_2, \longrightarrow_2)$ are **isomorphic**, denoted $T_1 \equiv_{iso} T_2$, if there exists a bijection $f : S_1 \rightarrow S_2$ such that

$$s \xrightarrow{\alpha}_1 t \quad \text{if and only if} \quad f(s) \xrightarrow{\alpha}_2 f(t).$$

It follows immediately that \equiv_{iso} is an equivalence. (Why?)

Example 10.3 (Abelian monoid laws for $+$ and \parallel)

For all CCS processes $P, Q \in Proc$,

1. $LTS(P + Q) \equiv_{iso} LTS(Q + P)$, $LTS(P \parallel Q) \equiv_{iso} LTS(Q \parallel P)$
2. $LTS((P + Q) + R) \equiv_{iso} LTS(P + (Q + R))$, $LTS((P \parallel Q) \parallel R) \equiv_{iso} LTS(P \parallel (Q \parallel R))$
3. $LTS(P + nil) \equiv_{iso} LTS(P \parallel nil) \equiv_{iso} LTS(P)$

Isomorphism II

Assumption

From now on, we will consider processes **modulo isomorphism**, i.e., we do not distinguish CCS processes with isomorphic LTSs.

Caveat

But: isomorphism is very **distinctive**. For instance,

$$X = a.X \quad \text{and} \quad Y = a.a.Y$$

are distinguished although both can (only) execute infinitely many *a*-actions and should thus be considered **equivalent**.

Requirements on Behavioural Equivalences

The Wish List for Behavioural Equivalences

1. **Less distinctive than isomorphism**: an equivalence should distinguish less processes than LTS isomorphism does, i.e., \equiv should be coarser than LTS isomorphism:

$$LTS(P) \equiv_{iso} LTS(Q) \implies P \equiv Q.$$

2. **More distinctive than trace equivalence**: an equivalence should distinguish more processes than trace equivalence does, i.e., \equiv should be finer than trace equivalence:

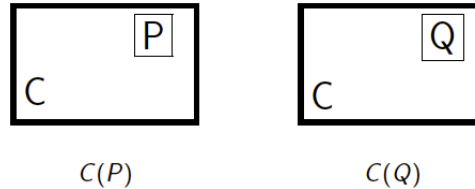
$$P \equiv Q \implies Tr(P) = Tr(Q).$$

3. **Congruence property**: the equivalence must be substitutive with respect to all CCS operators (see next slide).
4. **Deadlock preservation**: equivalent processes should have the same deadlock behaviour, i.e., equivalent process can either both deadlock, or both cannot.¹
5. Optional: the **coarsest** possible equivalence: there should be no less discriminating equivalence satisfying all these requirements.

¹Later, we will enlarge this to a set of properties that can be expressed in a logic.

Requirements on Behavioural Equivalences

What is a Congruence?



CCS contexts informally

A **CCS context** is a CCS process fragment with a “hole” in it (examples on the board).

CCS congruences informally

Relation \equiv is a **CCS congruence** whenever $P \equiv Q$ implies $C(P) \equiv C(Q)$ for every CCS context C .

Requirements on Behavioural Equivalences

The Importance of Congruences

CCS congruences informally

Relation \equiv is a **congruence** whenever $P \equiv Q$ implies $C(P) \equiv C(Q)$ for every CCS context C .

Example 10.4 (Congruence)

Let $a \equiv b$ for $a, b \in \mathbb{Z}$ whenever $a \bmod k = b \bmod k$, for some $k \in \mathbb{N}_+$.
Equivalence relation \equiv is a congruence for addition and multiplication.

Important motivations of requiring \equiv to be a congruence on processes:

1. **Model-based development through refinement**: replacing an abstract model *Spec* by a more detailed model *Impl*
2. **Abstraction/optimisation**: replacing a large (concrete) model *Impl* by a smaller (more abstract) model *Spec*.

Remark: congruences induce **quotient structures** with equivalence classes as elements

Requirements on Behavioural Equivalences

CCS Congruences Formally

Definition 10.5 (CCS congruence)

An equivalence relation $\equiv \subseteq \text{Prc} \times \text{Prc}$ is a **CCS congruence** if it is preserved by all CCS constructs, i.e., if $P, Q \in \text{Prc}$ with $P \equiv Q$ then:

$$\begin{aligned} \alpha.P &\equiv \alpha.Q && \text{for every } \alpha \in \text{Act} \\ P + R &\equiv Q + R && \text{for every } R \in \text{Prc} \\ P \parallel R &\equiv Q \parallel R && \text{for every } R \in \text{Prc} \\ P \setminus L &\equiv Q \setminus L && \text{for every } L \subseteq A \\ P[f] &\equiv Q[f] && \text{for every } f : A \rightarrow A \end{aligned}$$

Thus, a CCS congruence is **substitutive** for all possible CCS contexts.

Requirements on Behavioural Equivalences

Deadlocks

Definition 10.6 (Deadlock)

Let $P, Q \in Prc$ and $w \in Act^*$ such that $P \xrightarrow{w} Q$ and $Q \not\rightarrow$. Then Q is called a **w -deadlock** of P .

Example 10.7

$P = a.b.nil + a.nil$ has an a -deadlock, whereas $Q = a.b.nil$ has not.

Such properties are important as it can be crucial that a certain action is eventually possible.

Definition 10.8 (Deadlock sensitivity)

Relation $\equiv \subseteq Prc \times Prc$ is **deadlock sensitive** whenever:

$P \equiv Q$ implies $(\forall w \in Act^*. P \text{ has a } w\text{-deadlock iff } Q \text{ has a } w\text{-deadlock})$.

Trace Equivalence Revisited

Trace Equivalence

Trace language (Definition 3.2)

The **trace language** of $P \in Prc$ is defined by:

$$Tr(P) := \{w \in Act^* \mid \exists P' \in Prc. P \xrightarrow{w} P'\}.$$

Trace equivalence (Definition 3.2)

$P, Q \in Prc$ are called **trace equivalent** iff $Tr(P) = Tr(Q)$.

Trace equivalence is evidently an equivalence relation and is less discriminative than isomorphism.

Trace Equivalence Revisited

Trace Equivalence is a Congruence

Theorem 10.9

Trace equivalence is a CCS congruence.

Proof.

By structural induction over the syntax of CCS processes. For $+$ this proceeds as follows:

- Let $P, Q \in Prc$ with $Tr(P) = Tr(Q)$.
- Then for $R \in Prc$ it holds:

$$Tr(P + R) = Tr(P) \cup Tr(R) = Tr(Q) \cup Tr(R) = Tr(Q + R).$$

- Thus, $P + R$ and $Q + R$ are trace equivalent.

For the other CCS constructs, the proof goes along similar lines. Exercise: do the proof for \parallel . □

Trace Equivalence Revisited

Two coffee machines

Example 10.10

Consider the coffee/tea machines CTM and its variant CTM' :

$$CTM = coin. (\overline{coffee}.CTM + \overline{tea}.CTM)$$

$$CTM' = coin.\overline{coffee}.CTM' + coin.\overline{tea}.CTM'$$

Note the difference between the two processes. Nevertheless:

$$Tr(CTM) = Tr(CTM').$$

Are we satisfied? No, as CTM and CTM' differ in the context:

$$C(\cdot) = (\underbrace{\cdot}_{\text{hole}} \parallel CA) \setminus \{coin, coffee, tea\} \text{ with } CA = \overline{coin}.coffee.CA.$$

Why? $C(CTM')$ may yield a deadlock, but $C(CTM)$ does not.

Trace Equivalence Revisited

Checking Trace Equivalence

Traces by automata

For finite-state P , the trace language $Tr(P)$ of process P is accepted by the (non-deterministic) finite automaton obtained from the LTS of P with initial state P and making all states accepting (final).

Theorem 10.11

Checking trace equivalence of two finite processes is PSPACE-complete.

Proof.

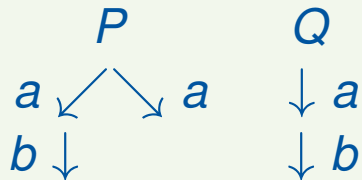
Checking whether $Tr(P) = Tr(Q)$, for finite-state P and Q , boils down to deciding whether their non-deterministic automata accept the same language. As this problem in automata theory is PSPACE-complete, it follows that checking $Tr(P) = Tr(Q)$ is PSPACE-complete. □

Trace Equivalence Revisited

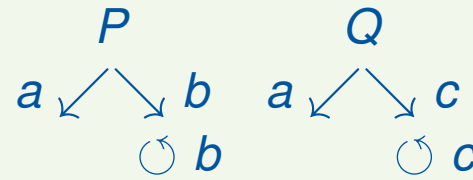
Traces and Deadlocks

Example 10.12 (Traces and deadlocks)

Traces and deadlocks are independent in the following sense:



same traces
different deadlocks



different traces
same deadlocks

But: processes with **finite trace sets** and identical deadlocks are trace equivalent (since every trace is a prefix of some deadlock).

Trace Equivalence Revisited

Summary: Trace Equivalence

1. Trace equivalence equates processes that have the same traces, i.e., action sequences
2. Isomorphism implies trace equivalence
3. Trace equivalence trivially implies trace equivalence
4. Trace equivalence is a CCS congruence
5. Trace equivalence is **not** deadlock sensitive.
6. Checking trace equivalence is PSPACE-complete

Other Forms of Trace Equivalence

Completed Trace Equivalence

Definition 10.13 (Completed traces)

A **completed trace** of $P \in \text{Prc}$ is a sequence $w \in \text{Act}^*$ such that:

$$P \xrightarrow{w} Q \quad \text{and} \quad Q \not\rightarrow$$

for some $Q \in \text{Prc}$.

The completed traces of process P may be seen as capturing its **deadlock behaviour**, as they are precisely the action sequences that could lead to a process from which no transition is possible (i.e., is a deadlock).

Exercise

Check whether completed trace equivalence is a congruence for restriction.

Other Forms of Trace Equivalence

Further Variations of Trace Equivalence

Definition 10.14 (Ready trace equivalence)

(Baeten et al.)

A sequence $A_0\alpha_0A_1\alpha_1 \dots \alpha_nA_n$ with $A_i \subseteq Act$ and $\alpha_i \in Act$ ($i \in \mathbb{N}$) is a **ready trace** of process P if $P = P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} P_n$ such that $A_i = \{\alpha \in Act \mid P_i \xrightarrow{\alpha}\}$. Processes P and Q are **ready-trace equivalent** if they have exactly the same set of ready traces.

Definition 10.15 (Failure trace equivalence)

(Reed and Roscoe)

A sequence $A_0\alpha_0A_1\alpha_1 \dots \alpha_nA_n$ with $A_i \subseteq Act$ and $\alpha_i \in Act$ ($i \in \mathbb{N}$) is a **failure trace** of process P if $P = P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} P_n$ such that $A_i \cap \{\alpha \in Act \mid P_i \xrightarrow{\alpha}\} = \emptyset$. Processes P and Q are **failure-trace equivalent** if they have exactly the same set of failure traces.

Example 10.16

$\alpha.P + \alpha.Q$ and $\alpha.P + \alpha.Q + \alpha.(P + Q)$ are failure-trace equivalent for every $P, Q \in Proc$ and $\alpha \in Act$, but not ready-trace equivalent

Summary

1. Behavioural equivalences should be
 - i. less distinctive than isomorphism
 - ii. more distinctive than trace equivalence
 - iii. a CCS congruence
 - iv. deadlock sensitive
2. Trace equivalence
 - i. equates processes that have the same traces, i.e., action sequences
 - ii. is a CCS congruence
 - iii. is **not** deadlock sensitive
 - iv. checking trace equivalence is PSPACE-complete
3. Variations: completed, ready, and failure traces