Static Program Analysis

Lecture 8: Dataflow Analysis VII (Narrowing & DFA with Conditional Branches)

Winter Semester 2016/17

Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/teaching/ws-1617/spa/

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Organisational Matters

Lecture Thu 01.12.2016 $\rightarrow$ **Tue 06.12.2016** **AH 2**

Outline of Lecture 8

Undecidability of the MOP Solution I

Theorem (Undecidability of MOP solution)

The MOP solution for Constant Propagation is undecidable.

Proof.

Based on undecidability of Modified Post Correspondence Problem (MPCP):
Let Γ be some alphabet, $n \in \mathbb{N}$, and $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Gamma^+$.
Do there exist $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with $m \geq 1$ and $i_1 = 1$ such that
$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$?

Undecidability of the MOP Solution I

Theorem (Undecidability of MOP solution)

The MOP solution for Constant Propagation is undecidable.

Proof.

Based on undecidability of Modified Post Correspondence Problem (MPCP):
Let Γ be some alphabet, $n \in \mathbb{N}$, and $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Gamma^+$.
Do there exist $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with $m \geq 1$ and $i_1 = 1$ such that
$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$?

Given a MPCP, we construct a WHILE program (with strings and Booleans) whose MOP analysis detects a constant property iff the MPCP has no solution (see next slide).

Undecidability of the MOP Solution II

Proof (continued).

```
x := u1;y := v1;
while ... do
  if ... then
    x := x ++ u1;
    y := y ++ v1
  else if ... then
    ⋮
  else
    x := x ++ un;
    y := y ++ vn
  end ... end
end;
~~if x = y then z := 1 else z := 0 end;~~
z := (x = y);
[skip]l
```

Undecidability of the MOP Solution II

Proof (continued).

```
x := u₁;y := v₁;
while ... do
  if ... then
    x := x ++ u₁;
    y := y ++ v₁
  else if ... then
    ⋮
  else
    x := x ++ uₙ;
    y := y ++ vₙ
  end ... end
end;
~~if x = y then z := 1 else z := 0 end;~~
z := (x = y);
[skip]^l
```

Then: $\mathrm{mop}(l)(\mathtt{z}) = \mathrm{false}$

Undecidability of the MOP Solution II

Proof (continued).

```
x := u1;y := v1;
while ... do
  if ... then
    x := x ++ u1;
    y := y ++ v1
  else if ... then
    ⋮
  else
    x := x ++ un;
    y := y ++ vn
  end ... end
end;
```
~~if x = y then z := 1 else z := 0 end;~~
```
z := (x = y);
[skip]^l
```

Then: $\mathrm{mop}(l)(\mathtt{z}) = \mathrm{false}$

$\iff \mathtt{x} \neq \mathtt{y}$ at the end of every path up to l

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Undecidability of the MOP Solution II

Proof (continued).

```
x := u1;y := v1;
while ... do
  if ... then
    x := x ++ u1;
    y := y ++ v1
  else if ... then
    ⋮
  else
    x := x ++ un;
    y := y ++ vn
  end ... end
end;
~~if x = y then z := 1 else z := 0 end;~~
z := (x = y);
[skip]^l
```

Then: $\mathrm{mop}(l)(\mathtt{z}) = \mathrm{false}$

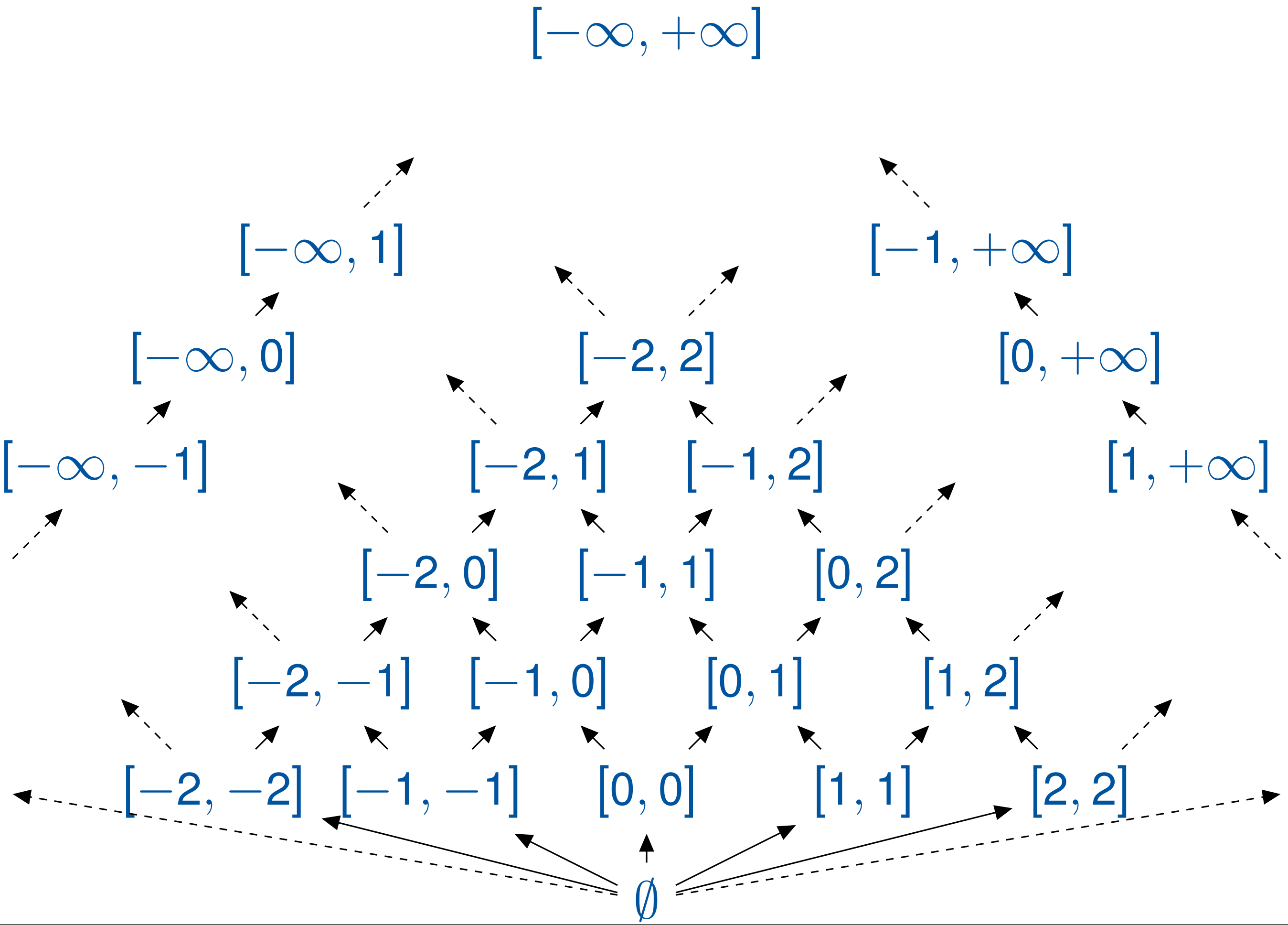
$\iff$ $\mathtt{x} \neq \mathtt{y}$ at the end of every path up to l

$\iff$ the MPCP has no solution

$\square$

Outline of Lecture 8

The Complete Lattice of Interval Analysis



Formalising Interval Analysis I

The dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ is given by

- set of labels $Lab := Lab_c$
- extremal labels $E := \{\text{init}(c)\}$ (forward problem)
- flow relation $F := \text{flow}(c)$ (forward problem)
- complete lattice $(D, \sqsubseteq)$ where
  - $D := \{\delta \mid \delta : Var_c \to Int\}$
  - $\delta_1 \sqsubseteq \delta_2$ iff $\delta_1(x) \subseteq \delta_2(x)$ for every $x \in Var_c$
- $\iota := \top_D : Var_c \to Int : x \mapsto \top_{Int}$ (with $\top_{Int} = [-\infty, +\infty]$)
- φ: see next slide

Formalising Interval Analysis II

Transfer functions $\{\varphi_l \mid l \in Lab\}$ are defined by

$$\varphi_l(\delta) := \begin{cases} \delta & \text{if } B^l = \texttt{skip} \text{ or } B^l \in BExp \\ \delta[x \mapsto val_\delta(a)] & \text{if } B^l = (x := a) \end{cases}$$

where

$$val_\delta(x) := \delta(x) \qquad val_\delta(a_1+a_2) := val_\delta(a_1) \oplus val_\delta(a_2)$$
$$val_\delta(z) := [z, z] \qquad val_\delta(a_1-a_2) := val_\delta(a_1) \ominus val_\delta(a_2)$$
$$val_\delta(a_1*a_2) := val_\delta(a_1) \odot val_\delta(a_2)$$

with

$$\emptyset \oplus J := J \oplus \emptyset := \emptyset \ominus J := \ldots := \emptyset$$
$$[y_1, y_2] \oplus [z_1, z_2] := [y_1 + z_1, y_2 + z_2]$$
$$[y_1, y_2] \ominus [z_1, z_2] := [y_1 - z_2, y_2 - z_1]$$
$$[y_1, y_2] \odot [z_1, z_2] := [\textstyle\bigsqcap\{y_1z_1, y_1z_2, y_2z_1, y_2z_2\}, \bigsqcup\{y_1z_1, y_1z_2, y_2z_1, y_2z_2\}]$$

Widening Operators

Definition (Widening operator)

Let $(D, \sqsubseteq)$ be a complete lattice. A mapping $\nabla : D \times D \to D$ is called widening operator if

- for every $d_1, d_2 \in D$,
$$d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2$$
  and
- for all ascending chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$, the ascending chain $d_0^\nabla \sqsubseteq d_1^\nabla \sqsubseteq \ldots$ eventually stabilises where
$$d_0^\nabla := d_0 \text{ and } d_{i+1}^\nabla := d_i^\nabla \nabla d_{i+1} \text{ for each } i \in \mathbb{N}$$

Remarks:

- $(d_i^\nabla)_{i\in\mathbb{N}}$ is clearly an ascending chain as $d_{i+1}^\nabla = d_i^\nabla \nabla d_{i+1} \sqsupseteq d_i^\nabla \sqcup d_{i+1} \sqsupseteq d_i^\nabla$
- In contrast to $\sqcup$, ∇ does *not* have to be commutative, associative, monotonic, nor absorptive ($d \nabla d = d$)
- The requirement $d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2$ guarantees soundness of widening

Applying Widening to Interval Analysis

- A widening operator: $\nabla : \mathit{Int} \times \mathit{Int} \to \mathit{Int}$ with

$$
\begin{aligned}
\emptyset \nabla J &:= J \nabla \emptyset := J \\
[x_1, x_2] \nabla [y_1, y_2] &:= [z_1, z_2] \quad \text{where} \\
z_1 &:= \begin{cases} x_1 & \text{if } x_1 \le y_1 \\ -\infty & \text{otherwise} \end{cases} \\
z_2 &:= \begin{cases} x_2 & \text{if } x_2 \ge y_2 \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

- Widening turns infinite ascending chain

$$J_0 = \emptyset \subseteq J_1 = [1, 1] \subseteq J_2 = [1, 2] \subseteq J_3 = [1, 3] \subseteq \ldots$$

  into a finite one:

$$
\begin{aligned}
J_0^\nabla &= J_0 = \emptyset \\
J_1^\nabla &= J_0^\nabla \nabla J_1 = \emptyset \nabla [1, 1] = [1, 1] \\
J_2^\nabla &= J_1^\nabla \nabla J_2 = [1, 1] \nabla [1, 2] = [1, +\infty] \\
J_3^\nabla &= J_2^\nabla \nabla J_3 = [1, +\infty] \nabla [1, 3] = [1, +\infty]
\end{aligned}
$$

- In fact, the maximal chain size arising with this operator is 4:

$$\emptyset \subseteq [3, 7] \subseteq [3, +\infty] \subseteq [-\infty, +\infty]$$

Worklist Algorithm with Widening

Goal: extend Algorithm 5.1 by widening to ensure termination

Algorithm (Worklist algorithm)

Input: *dataflow system* $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$

Variables: $W \in (Lab \times Lab)^*$, $\{\mathrm{AI}_l \in D \mid l \in Lab\}$

Procedure: $W := \varepsilon$; **for** $(l, l') \in F$ **do** $W := W \cdot (l, l')$; *% Initialise W*
    for $l \in Lab$ **do**
        if $l \in E$ **then** $\mathrm{AI}_l := \iota$ **else** $\mathrm{AI}_l := \perp_D$; *% Initialise AI*
    while $W \neq \varepsilon$ **do**
        $(l, l') :=$ **head**(W); $W :=$ **tail**(W); *% Next control-flow edge*
        if $\varphi_l(\mathrm{AI}_l) \not\sqsubseteq \mathrm{AI}_{l'}$ **then** *% Fixpoint not yet reached*
            $\mathrm{AI}_{l'} := \mathrm{AI}_{l'} \sqcup \varphi_l(\mathrm{AI}_l)$; *% Update analysis information*
            for $(l', l'') \in F$ **do**
                if (l', l'') *not in* W **then** $W := (l', l'') \cdot W$; *% Propagate modification*

Output: $\{\mathrm{AI}_l \mid l \in Lab\}$

Worklist Algorithm with Widening

Goal: extend Algorithm 5.1 by widening to ensure termination

Algorithm (Worklist algorithm with widening)

Input: *dataflow system* $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$

Variables: $W \in (Lab \times Lab)^*$, $\{\mathrm{AI}_l \in D \mid l \in Lab\}$

Procedure: $W := \varepsilon$; **for** $(l, l') \in F$ **do** $W := W \cdot (l, l')$; *% Initialise W*
for $l \in Lab$ **do**
if $l \in E$ **then** $\mathrm{AI}_l := \iota$ **else** $\mathrm{AI}_l := \bot_D$; *% Initialise AI*
while $W \neq \varepsilon$ **do**
$(l, l') :=$ **head**(W); $W :=$ **tail**(W); *% Next control-flow edge*
if $\varphi_l(\mathrm{AI}_l) \not\sqsubseteq \mathrm{AI}_{l'}$ **then** *% Fixpoint not yet reached*
$\mathrm{AI}_{l'} := \mathrm{AI}_{l'} \nabla \varphi_l(\mathrm{AI}_l)$; *% Update analysis information*
for $(l', l'') \in F$ **do**
if (l', l'') *not in* W **then** $W := (l', l'') \cdot W$; *% Propagate modification*

Output: $\{\mathrm{AI}_l \mid l \in Lab\}$, *denoted by* $\mathrm{fix}^{\nabla}(\Phi_S)$

Worklist Algorithm with Widening

Goal: extend Algorithm 5.1 by widening to ensure termination

Algorithm (Worklist algorithm with widening)

Input: *dataflow system* $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$

Variables: $W \in (Lab \times Lab)^*$, $\{\mathrm{AI}_l \in D \mid l \in Lab\}$

Procedure: $W := \varepsilon$; **for** $(l, l') \in F$ **do** $W := W \cdot (l, l')$; *% Initialise W*
 for $l \in Lab$ **do**
  if $l \in E$ **then** $\mathrm{AI}_l := \iota$ **else** $\mathrm{AI}_l := \bot_D$; *% Initialise AI*
 while $W \neq \varepsilon$ **do**
  $(l, l') :=$ **head**(W); $W :=$ **tail**(W); *% Next control-flow edge*
  if $\varphi_l(\mathrm{AI}_l) \not\sqsubseteq \mathrm{AI}_{l'}$ **then** *% Fixpoint not yet reached*
   $\mathrm{AI}_{l'} := \mathrm{AI}_{l'} \nabla \varphi_l(\mathrm{AI}_l)$; *% Update analysis information*
   for $(l', l'') \in F$ **do**
    if (l', l'') *not in* W **then** $W := (l', l'') \cdot W$; *% Propagate modification*

Output: $\{\mathrm{AI}_l \mid l \in Lab\}$, *denoted by* $\mathrm{fix}^\nabla(\Phi_S)$

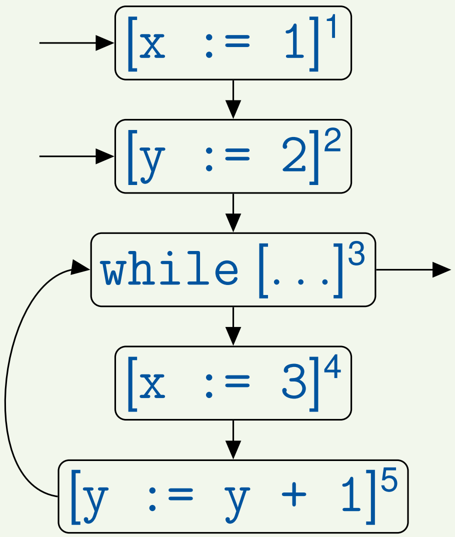
Remark: due to widening, only $\mathrm{fix}^\nabla(\Phi_S) \sqsupseteq \mathrm{fix}(\Phi_S)$ is guaranteed (cf. Thm. 5.4)

Outline of Lecture 8

Narrowing

Another Widening Example

Example 8.1

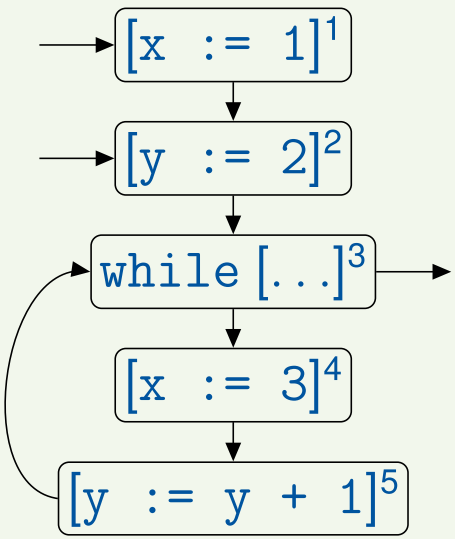


Transfer functions (for $\delta = (J_x, J_y)$):

$$
\begin{aligned}
\varphi_1(J_x, J_y) &= ([1,1], J_y) \\
\varphi_2(J_x, J_y) &= (J_x, [2,2]) \\
\varphi_3(J_x, J_y) &= (J_x, J_y) \\
\varphi_4(J_x, J_y) &= ([3,3], J_y) \\
\varphi_5(J_x, \emptyset) &= (J_x, \emptyset) \\
\varphi_5(J_x, [y_1, y_2]) &= (J_x, [y_1 + 1, y_2 + 1])
\end{aligned}
$$

Another Widening Example

Example 8.1



Transfer functions (for $\delta = (J_\mathtt{x}, J_\mathtt{y})$):

$$
\begin{aligned}
\varphi_1(J_\mathtt{x}, J_\mathtt{y}) &= ([1,1], J_\mathtt{y}) \\
\varphi_2(J_\mathtt{x}, J_\mathtt{y}) &= (J_\mathtt{x}, [2,2]) \\
\varphi_3(J_\mathtt{x}, J_\mathtt{y}) &= (J_\mathtt{x}, J_\mathtt{y}) \\
\varphi_4(J_\mathtt{x}, J_\mathtt{y}) &= ([3,3], J_\mathtt{y}) \\
\varphi_5(J_\mathtt{x}, \emptyset) &= (J_\mathtt{x}, \emptyset) \\
\varphi_5(J_\mathtt{x}, [y_1, y_2]) &= (J_\mathtt{x}, [y_1 + 1, y_2 + 1])
\end{aligned}
$$

Application of worklist algorithm

1. without widening (omitted): diverges (for $\mathtt{y}$) with expected result for $\mathtt{x}$: $\mathsf{AI}_3(\mathtt{x}) = [1, 3]$
2. with widening (on the board): terminates with unexpected result for $\mathtt{x}$: $\mathsf{AI}_3(\mathtt{x}) = [1, +\infty]$

Idea of Narrowing

- **Observation:** widening can "shoot above the target", i.e., lead to unnecessarily imprecise results

Narrowing

Idea of Narrowing

- **Observation:** widening can "shoot above the target", i.e., lead to unnecessarily imprecise results
- **Solution:** improvement by iterating again from the result obtained by widening (i.e., from $\text{fix}^{\nabla}(\Phi_S)$)

$$\implies \text{compute } \Phi_S^k(\text{fix}^{\nabla}(\Phi_S)) \text{ for } k = 1, 2, \ldots$$

Narrowing

Idea of Narrowing

- **Observation:** widening can “shoot above the target”, i.e., lead to unnecessarily imprecise results
- **Solution:** improvement by iterating again from the result obtained by widening (i.e., from $\mathrm{fix}^{\nabla}(\Phi_S)$)

$$\Longrightarrow \text{ compute } \Phi_S^k(\mathrm{fix}^{\nabla}(\Phi_S)) \text{ for } k = 1, 2, \ldots$$

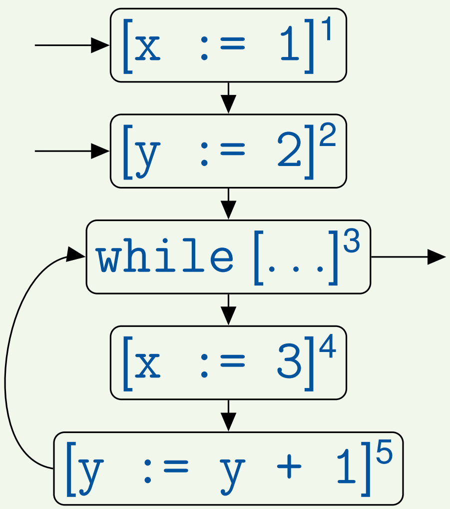
- **Soundness:** $\mathrm{fix}^{\nabla}(\Phi_S) \sqsupseteq \mathrm{fix}(\Phi_S)$ (cf. Alg. 7.5)

$$\Longrightarrow \Phi_S^k(\mathrm{fix}^{\nabla}(\Phi_S)) \sqsupseteq \Phi_S^k(\mathrm{fix}(\Phi_S)) = \mathrm{fix}(\Phi_S)$$

(since Φ_S and thus Φ_S^k monotonic)

Narrowing Example

Example 8.2 (cf. Example 8.1)



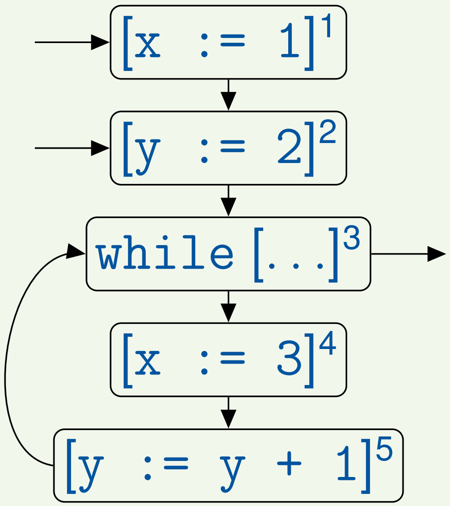
Transfer functions (for $\delta = (J_\mathrm{x}, J_\mathrm{y})$):

$$
\begin{aligned}
\varphi_1(J_\mathrm{x}, J_\mathrm{y}) &= ([1,1], J_\mathrm{y}) \\
\varphi_2(J_\mathrm{x}, J_\mathrm{y}) &= (J_\mathrm{x}, [2,2]) \\
\varphi_3(J_\mathrm{x}, J_\mathrm{y}) &= (J_\mathrm{x}, J_\mathrm{y}) \\
\varphi_4(J_\mathrm{x}, J_\mathrm{y}) &= ([3,3], J_\mathrm{y}) \\
\varphi_5(J_\mathrm{x}, \emptyset) &= (J_\mathrm{x}, \emptyset) \\
\varphi_5(J_\mathrm{x}, [y_1, y_2]) &= (J_\mathrm{x}, [y_1 + 1, y_2 + 1])
\end{aligned}
$$

Narrowing

Narrowing Example

Example 8.2 (cf. Example 8.1)



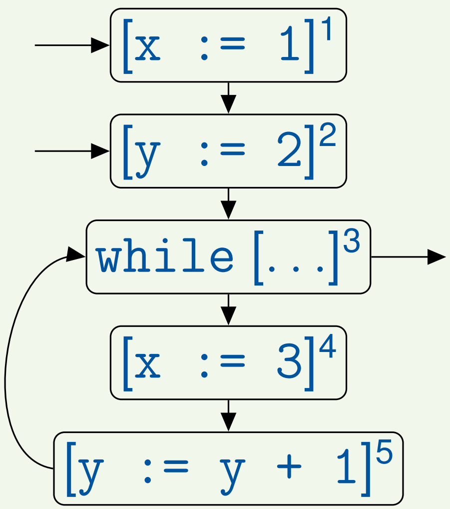
Transfer functions (for $\delta = (J_x, J_y)$):

$$
\begin{aligned}
\varphi_1(J_x, J_y) &= ([1,1], J_y) \\
\varphi_2(J_x, J_y) &= (J_x, [2,2]) \\
\varphi_3(J_x, J_y) &= (J_x, J_y) \\
\varphi_4(J_x, J_y) &= ([3,3], J_y) \\
\varphi_5(J_x, \emptyset) &= (J_x, \emptyset) \\
\varphi_5(J_x, [y_1, y_2]) &= (J_x, [y_1 + 1, y_2 + 1])
\end{aligned}
$$

Narrowing	AI_1	AI_2	AI_3	AI_4	AI_5
$\mathsf{fix}^{\nabla}(\Phi_S)$	$(\top, \top)$	$([1,1], \top)$	$([1, +\infty], [2, +\infty])$	$([1, +\infty], [2, +\infty])$	$([3,3], [2, +\infty])$

Narrowing Example

Example 8.2 (cf. Example 8.1)



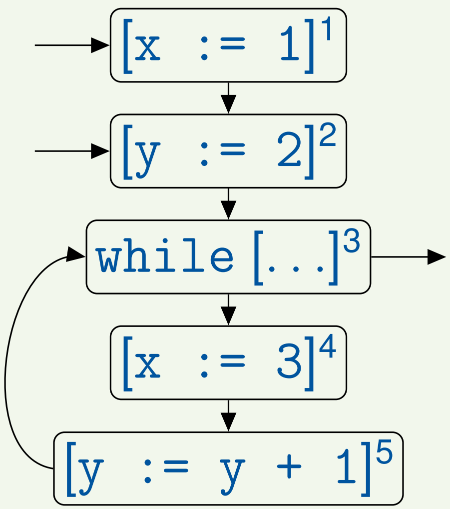
Transfer functions (for $\delta = (J_x, J_y)$):

$$
\begin{aligned}
\varphi_1(J_x, J_y) &= ([1,1], J_y)\\
\varphi_2(J_x, J_y) &= (J_x, [2,2])\\
\varphi_3(J_x, J_y) &= (J_x, J_y)\\
\varphi_4(J_x, J_y) &= ([3,3], J_y)\\
\varphi_5(J_x, \emptyset) &= (J_x, \emptyset)\\
\varphi_5(J_x, [y_1, y_2]) &= (J_x, [y_1+1, y_2+1])
\end{aligned}
$$

Narrowing	AI_1	AI_2	AI_3	AI_4	AI_5
$\mathrm{fix}^\nabla(\Phi_S)$	$(\top, \top)$	$([1,1], \top)$	$([1,+\infty], [2,+\infty])$	$([1,+\infty], [2,+\infty])$	$([3,3], [2,+\infty])$
$\Phi_S(\mathrm{fix}^\nabla(\Phi_S))$	$(\top, \top)$	$([1,1], \top)$	$([1,3], [2,+\infty])$	$([1,+\infty], [2,+\infty])$	$([3,3], [2,+\infty])$

Narrowing Example

Example 8.2 (cf. Example 8.1)



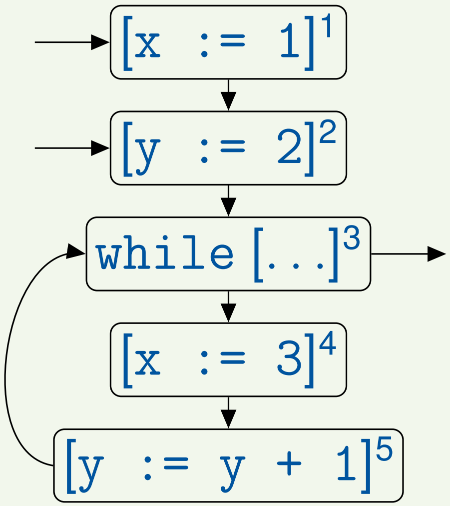
Transfer functions (for $\delta = (J_{\mathrm{x}}, J_{\mathrm{y}})$):

$$\begin{aligned}
\varphi_1(J_{\mathrm{x}}, J_{\mathrm{y}}) &= ([1,1], J_{\mathrm{y}})\\
\varphi_2(J_{\mathrm{x}}, J_{\mathrm{y}}) &= (J_{\mathrm{x}}, [2,2])\\
\varphi_3(J_{\mathrm{x}}, J_{\mathrm{y}}) &= (J_{\mathrm{x}}, J_{\mathrm{y}})\\
\varphi_4(J_{\mathrm{x}}, J_{\mathrm{y}}) &= ([3,3], J_{\mathrm{y}})\\
\varphi_5(J_{\mathrm{x}}, \emptyset) &= (J_{\mathrm{x}}, \emptyset)\\
\varphi_5(J_{\mathrm{x}}, [y_1, y_2]) &= (J_{\mathrm{x}}, [y_1 + 1, y_2 + 1])
\end{aligned}$$

Narrowing	AI_1	AI_2	AI_3	AI_4	AI_5
$\mathrm{fix}^{\nabla}(\Phi_S)$	$(\top, \top)$	$([1,1], \top)$	$([1,+\infty], [2,+\infty])$	$([1,+\infty], [2,+\infty])$	$([3,3], [2,+\infty])$
$\Phi_S(\mathrm{fix}^{\nabla}(\Phi_S))$	$(\top, \top)$	$([1,1], \top)$	$([1,3], [2,+\infty])$	$([1,+\infty], [2,+\infty])$	$([3,3], [2,+\infty])$
$\Phi_S^2(\mathrm{fix}^{\nabla}(\Phi_S))$	$(\top, \top)$	$([1,1], \top)$	$([1,3], [2,+\infty])$	$([1,3], [2,+\infty])$	$([3,3], [2,+\infty])$

Narrowing Example

Example 8.2 (cf. Example 8.1)



Transfer functions (for $\delta = (J_\mathrm{x}, J_\mathrm{y})$):

$$\begin{aligned}
\varphi_1(J_\mathrm{x}, J_\mathrm{y}) &= ([1,1], J_\mathrm{y})\\
\varphi_2(J_\mathrm{x}, J_\mathrm{y}) &= (J_\mathrm{x}, [2,2])\\
\varphi_3(J_\mathrm{x}, J_\mathrm{y}) &= (J_\mathrm{x}, J_\mathrm{y})\\
\varphi_4(J_\mathrm{x}, J_\mathrm{y}) &= ([3,3], J_\mathrm{y})\\
\varphi_5(J_\mathrm{x}, \emptyset) &= (J_\mathrm{x}, \emptyset)\\
\varphi_5(J_\mathrm{x}, [y_1, y_2]) &= (J_\mathrm{x}, [y_1+1, y_2+1])
\end{aligned}$$

Narrowing	AI_1	AI_2	AI_3	AI_4	AI_5
$\mathrm{fix}^\nabla(\Phi_S)$	$(\top,\top)$	$([1,1],\top)$	$([1,+\infty],[2,+\infty])$	$([1,+\infty],[2,+\infty])$	$([3,3],[2,+\infty])$
$\Phi_S(\mathrm{fix}^\nabla(\Phi_S))$	$(\top,\top)$	$([1,1],\top)$	$([1,3],[2,+\infty])$	$([1,+\infty],[2,+\infty])$	$([3,3],[2,+\infty])$
$\Phi_S^2(\mathrm{fix}^\nabla(\Phi_S))$	$(\top,\top)$	$([1,1],\top)$	$([1,3],[2,+\infty])$	$([1,3],[2,+\infty])$	$([3,3],[2,+\infty])$
$\Phi_S^3(\mathrm{fix}^\nabla(\Phi_S))$	$(\top,\top)$	$([1,1],\top)$	$([1,3],[2,+\infty])$	$([1,3],[2,+\infty])$	$([3,3],[2,+\infty])$

Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)

Narrowing

Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)
- **But:** possible to stop after each step without losing soundness ($\Phi_S^k(\mathrm{fix}^\nabla(\Phi_S)) \sqsupseteq \mathrm{fix}(\Phi_S)$)

Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)
- **But:** possible to stop after each step without losing soundness ($\Phi_S^k(\text{fix}^\nabla(\Phi_S)) \sqsupseteq \text{fix}(\Phi_S)$)
- **In practice:** termination often ensured by using narrowing operators
  ($\approx$ counterpart of widening operator; definition omitted)

Outline of Lecture 8

Revision: Undecidability of the MOP Solution

Recap: Interval Analysis

Narrowing

Taking Conditional Branches into Account

Constant Propagation Analysis with Assertions

Taking Conditional Branches into Account I

- **So far:** values of conditions have been ignored in analysis
- Essentially: `if` and `while` statements treated as nondeterministic choice between the two branches

Taking Conditional Branches into Account I

- **So far:** values of conditions have been ignored in analysis
- Essentially: `if` and `while` statements treated as nondeterministic choice between the two branches

Example 8.3

```
y := 0;
z := 0;
while [x > 0]^l do
  if y < 17 then
    y := y + 1
  end;
  z := z + x;
  x := x - 1
end;
```

Taking Conditional Branches into Account I

- **So far:** values of conditions have been ignored in analysis
- Essentially: `if` and `while` statements treated as nondeterministic choice between the two branches

Example 8.3

```
y := 0;
z := 0;
while [x > 0]^l do
  if y < 17 then
    y := y + 1
  end;
  z := z + x;
  x := x - 1
end;
```

- Interval analysis (with widening) yields for l:

$$\mathtt{x} \in [-\infty, +\infty]$$
$$\mathtt{y} \in [0, +\infty]$$
$$\mathtt{z} \in [-\infty, +\infty]$$

Taking Conditional Branches into Account I

- **So far:** values of conditions have been ignored in analysis
- Essentially: `if` and `while` statements treated as nondeterministic choice between the two branches

Example 8.3

```
y := 0;
z := 0;
while [x > 0]^l do
  if y < 17 then
    y := y + 1
  end;
  z := z + x;
  x := x - 1
end;
```

- Interval analysis (with widening) yields for l:

$$\begin{aligned} x &\in [-\infty, +\infty] \\ y &\in [0, +\infty] \\ z &\in [-\infty, +\infty] \end{aligned}$$

- Too pessimistic! In fact,

$$\begin{aligned} x &\in [-\infty, +\infty] \\ y &\in [0, 17] \\ z &\in [0, +\infty] \end{aligned}$$

Taking Conditional Branches into Account II

- **Solution:** introduce transfer functions for branches

Taking Conditional Branches into Account II

- **Solution:** introduce transfer functions for branches
- **First approach:** attach (negated) conditions as labels to control flow edges
  - advantage: no language modification required
  - disadvantage: entails extension of DFA framework
  - will not further be considered here

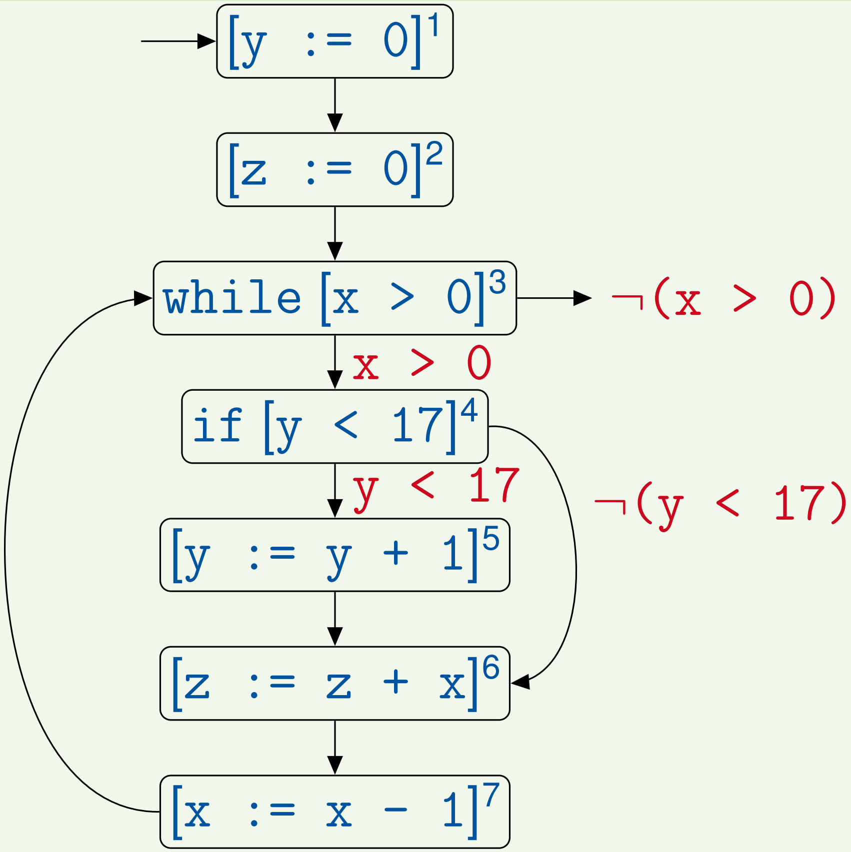
Taking Conditional Branches into Account II

- **Solution:** introduce transfer functions for branches
- **First approach:** attach (negated) conditions as labels to control flow edges
  - advantage: no language modification required
  - disadvantage: entails extension of DFA framework
  - will not further be considered here
- **Second approach:** encode conditions as assertions (proper statements)
  - advantage: DFA framework can be reused
  - disadvantage: entails extension of WHILE language
  - the way we will follow

Taking Conditional Branches into Account

Conditions as Edge Labels

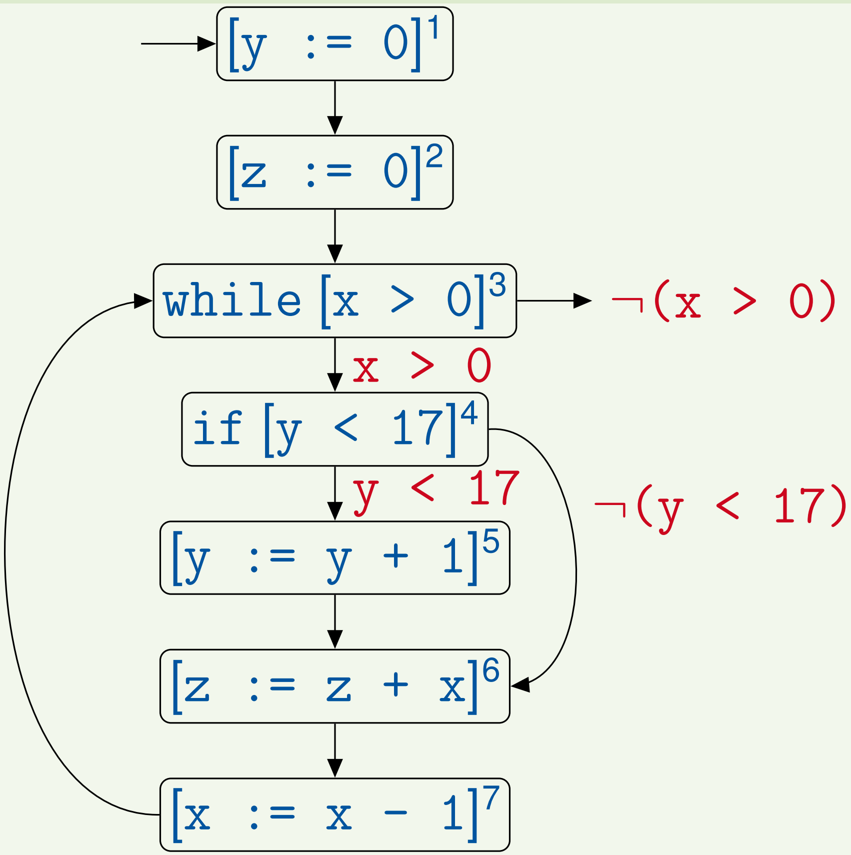
Example 8.4 (cf. Example 8.3)



Taking Conditional Branches into Account

Conditions as Edge Labels vs. Conditions as Assertions

Example 8.4 (cf. Example 8.3)




```
y := 0;
z := 0;
while x > 0 do
  assert x > 0;
  if y < 17 then
    assert y < 17;
    y := y + 1
  end;
  z := z + x;
  x := x - 1
end;
assert ¬(x > 0);
```

Taking Conditional Branches into Account

Extending the Syntax of WHILE Programs

Definition 8.5 (Labelled WHILE programs with assertions)

The syntax of labelled WHILE programs with assertions is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in AExp \\
b &::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\mathtt{skip}]^l \mid [x := a]^l \mid c_1 ; c_2 \mid \\
&\qquad \mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \mathtt{end} \mid \mathtt{while}\ [b]^l\ \mathtt{do}\ c\ \mathtt{end} \mid [\mathtt{assert}\ b]^l \in Cmd
\end{aligned}
$$

Extending the Syntax of WHILE Programs

Definition 8.5 (Labelled WHILE programs with assertions)

The syntax of labelled WHILE programs with assertions is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in AExp \\
b &::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1 ; c_2 \mid \\
&\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end} \mid \texttt{while } [b]^l \texttt{ do } c \texttt{ end} \mid [\texttt{assert } b]^l \in Cmd
\end{aligned}
$$

To be done:

- Definition of transfer functions for `assert` blocks (depending on analysis problem)
- Idea: assertions as filters that let only “compatible” information pass

Outline of Lecture 8

Constant Propagation Analysis with Assertions

Original Constant Propagation Analysis

So far:

- complete lattice $(D, \sqsubseteq)$ where
  - $D := \{\delta \mid \delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}\}$
    - $\delta(x) = z \in \mathbb{Z}$: x has constant value z (i.e., possible values in $\{z\}$)
    - $\delta(x) = \bot$: x undefined (i.e., possible values in $\emptyset$)
    - $\delta(x) = \top$: x overdefined (i.e., possible values in $\mathbb{Z}$)
  - $\sqsubseteq \subseteq D \times D$ defined by pointwise extension of $\bot \sqsubseteq z \sqsubseteq \top$ (for every $z \in \mathbb{Z}$)
- transfer functions $\{\varphi_l \mid l \in Lab\}$ defined by

$$\varphi_l(\delta) := \begin{cases} \delta & \text{if } B^l = \texttt{skip} \text{ or } B^l \in BExp \\ \delta[x \mapsto val_\delta(a)] & \text{if } B^l = (x := a) \end{cases}$$

where

$$val_\delta(x) := \delta(x) \qquad val_\delta(a_1\ op\ a_2) := \begin{cases} z_1\ op\ z_2 & \text{if } z_1, z_2 \in \mathbb{Z} \\ \bot & \text{if } z_1 = \bot \text{ or } z_2 = \bot \\ \top & \text{otherwise} \end{cases}$$
$$val_\delta(z) := z$$

for $z_1 := val_\delta(a_1)$ and $z_2 := val_\delta(a_2)$

Constant Propagation Analysis with Assertions

Transfer Functions of Assertions I

Additionally for $B^l = (\texttt{assert}\ b)$, $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$ and $x \in Var_c$:

$$\varphi_l(\delta)(x) := \begin{cases} \bot & \text{if } \nexists \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \\ z & \text{if } \forall \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \implies \sigma(x) = z \\ \top & \text{otherwise} \end{cases}$$

Constant Propagation Analysis with Assertions

Transfer Functions of Assertions I

Additionally for $B^l = (\texttt{assert}\ b)$, $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$ and $x \in Var_c$:

$$\varphi_l(\delta)(x) := \begin{cases} \bot & \text{if } \nexists \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \\ z & \text{if } \forall \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \implies \sigma(x) = z \\ \top & \text{otherwise} \end{cases}$$

where

- the set of δ-states is given by

$$\Sigma_\delta := \left\{ \sigma : Var_c \to \mathbb{Z} \,\middle|\, \forall y \in Var_c : \sigma(y) \in \begin{cases} \emptyset & \text{if } \delta(y) = \bot \\ \{z\} & \text{if } \delta(y) = z \\ \mathbb{Z} & \text{if } \delta(y) = \top \end{cases} \right\}$$

(and thus $\Sigma_\delta = \emptyset$ iff $\delta(y) = \bot$ for some $y \in Var_c$)

Transfer Functions of Assertions I

Additionally for $B^l = (\texttt{assert}\ b)$, $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$ and $x \in Var_c$:

$$\varphi_l(\delta)(x) := \begin{cases} \bot & \text{if } \nexists \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \\ z & \text{if } \forall \sigma \in \Sigma_\delta : val_\sigma(b) = \text{true} \implies \sigma(x) = z \\ \top & \text{otherwise} \end{cases}$$

where

- the set of δ-states is given by

$$\Sigma_\delta := \left\{ \sigma : Var_c \to \mathbb{Z} \,\middle|\, \forall y \in Var_c : \sigma(y) \in \begin{cases} \emptyset & \text{if } \delta(y) = \bot \\ \{z\} & \text{if } \delta(y) = z \\ \mathbb{Z} & \text{if } \delta(y) = \top \end{cases} \right\}$$

  (and thus $\Sigma_\delta = \emptyset$ iff $\delta(y) = \bot$ for some $y \in Var_c$)
- the evaluation function $val_\sigma : BExp \to \mathbb{B}$ for $\sigma : Var_c \to \mathbb{Z}$ is defined by

$$val_\sigma(t) := t$$

$$val_\sigma(a_1{=}a_2) := (val_\sigma(a_1) = val_\sigma(a_2))$$

($val_\sigma : AExp \to \mathbb{Z}$ on previous slide)

$$val_\sigma(\neg b) := \begin{cases} \text{true} & \text{if } val_\sigma(b) = \text{false} \\ \text{false} & \text{otherwise} \end{cases}$$

$$val_\sigma(b_1 \wedge b_2) := \begin{cases} \text{true} & \text{if } val_\sigma(b_1) = val_\sigma(b_2) = \text{true} \\ \text{false} & \text{otherwise} \end{cases}$$

Transfer Functions of Assertions II

Example 8.6

1. $\mathit{Var}_c = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$, $\delta = (\underbrace{\bot}_{\mathtt{x}}, \underbrace{1}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$
   $\implies \Sigma_\delta = \emptyset \implies \varphi_{\mathtt{assert}\ b}(\delta) = (\bot, \bot, \bot)$ for every $b \in \mathit{BExp}$

Constant Propagation Analysis with Assertions

Transfer Functions of Assertions II

Example 8.6

1. $\mathit{Var}_c = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$, $\delta = (\underbrace{\bot}_{\mathtt{x}}, \underbrace{1}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$

   $\implies \Sigma_\delta = \emptyset \implies \varphi_{\mathtt{assert}\ b}(\delta) = (\bot, \bot, \bot)$ for every $b \in \mathit{BExp}$

2. $\mathit{Var}_c = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$, $\delta = (\underbrace{1}_{\mathtt{x}}, \underbrace{2}_{\mathtt{y}}, \underbrace{\top}_{\mathtt{z}})$

   $$\implies \Sigma_\delta = \{(1, 2, z) \mid z \in \mathbb{Z}\} \implies \begin{aligned} \varphi_{\mathtt{assert\ x=y}}(\delta) &= (\bot, \bot, \bot) \\ \varphi_{\mathtt{assert\ y=z}}(\delta) &= (1, 2, 2) \\ \varphi_{\mathtt{assert\ y<z}}(\delta) &= (1, 2, \top) \\ \varphi_{\mathtt{assert\ x<=z} \wedge \mathtt{y>z}}(\delta) &= (1, 2, 1) \end{aligned}$$

Transfer Functions of Assertions II

Example 8.6

1. $Var_c = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$, $\delta = (\underbrace{\bot}_{\mathrm{x}}, \underbrace{1}_{\mathrm{y}}, \underbrace{\top}_{\mathrm{z}})$

   $\implies \Sigma_\delta = \emptyset \implies \varphi_{\mathtt{assert}\ b}(\delta) = (\bot, \bot, \bot)$ for every $b \in BExp$

2. $Var_c = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$, $\delta = (\underbrace{1}_{\mathrm{x}}, \underbrace{2}_{\mathrm{y}}, \underbrace{\top}_{\mathrm{z}})$

   $$\implies \Sigma_\delta = \{(1, 2, z) \mid z \in \mathbb{Z}\} \implies \begin{array}{r} \varphi_{\mathtt{assert\ x=y}}(\delta) = (\bot, \bot, \bot) \\ \varphi_{\mathtt{assert\ y=z}}(\delta) = (1, 2, 2) \\ \varphi_{\mathtt{assert\ y<z}}(\delta) = (1, 2, \top) \\ \varphi_{\mathtt{assert\ x<=z \wedge y>z}}(\delta) = (1, 2, 1) \end{array}$$

3. $Var_c = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$, $\delta = (\underbrace{1}_{\mathrm{x}}, \underbrace{\top}_{\mathrm{y}}, \underbrace{\top}_{\mathrm{z}})$

   $$\implies \Sigma_\delta = \{(1, z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}\} \implies \begin{array}{r} \varphi_{\mathtt{assert\ x=y}}(\delta) = (1, 1, \top) \\ \varphi_{\mathtt{assert\ y=z}}(\delta) = (1, \top, \top) \end{array}$$

Transfer Functions of Assertions III

Remarks:

- For $B^l = (\texttt{assert}\ b)$ and $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")

Constant Propagation Analysis with Assertions

Transfer Functions of Assertions III

Remarks:

- For $B^l = (\texttt{assert}\ b)$ and $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
- Constant propagation captures interdependencies between variables only when both are constant (cf. "`assert y=z`" in Example 8.6(3))

Constant Propagation Analysis with Assertions

Transfer Functions of Assertions III

Remarks:

- For $B^l = (\texttt{assert } b)$ and $\delta : \mathit{Var}_c \to \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
- Constant propagation captures interdependencies between variables only when both are constant (cf. "`assert y=z`" in Example 8.6(3))
- $\varphi_l(\delta)$ can be determined (or at least approximated) by Satisfiability Modulo Theories (SMT) techniques

Transfer Functions of Assertions III

Remarks:

- For $B^l = (\texttt{assert } b)$ and $\delta : Var_c \to \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
- Constant propagation captures interdependencies between variables only when both are constant (cf. "`assert y=z`" in Example 8.6(3))
- $\varphi_l(\delta)$ can be determined (or at least approximated) by Satisfiability Modulo Theories (SMT) techniques
- If $\mathrm{CP}_l(x) = \bot$ for some $l \in Lab_c$ and $x \in Var_c$, then l is unreachable (and $\mathrm{CP}_l(y) = \bot$ for all $y \in Var_c$)

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]¹ then
  [assert x = 1]²;
  [y := x + 1]³
else
  [assert ¬(x = 1)]⁴;
  [y := 2]⁵
end;
[skip]⁶
```

Without assertions	**With assertions**
$\mathrm{CP}_1 = (\top, \top)$	$\mathrm{CP}_1 = (\top, \top)$

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Without assertions	With assertions
$CP_1 = (\top, \top)$	$CP_1 = (\top, \top)$
	$CP_2 = (\top, \top)$

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Without assertions	With assertions
$CP_1 = (\top, \top)$	$CP_1 = (\top, \top)$
	$CP_2 = (\top, \top)$
$CP_3 = (\top, \top)$	$CP_3 = (1, \top)$

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Without assertions	With assertions
$CP_1 = (\top, \top)$	$CP_1 = (\top, \top)$
	$CP_2 = (\top, \top)$
$CP_3 = (\top, \top)$	$CP_3 = (1, \top)$
	$CP_4 = (\top, \top)$

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Without assertions	With assertions
$CP_1 = (\top, \top)$	$CP_1 = (\top, \top)$
	$CP_2 = (\top, \top)$
$CP_3 = (\top, \top)$	$CP_3 = (1, \top)$
	$CP_4 = (\top, \top)$
$CP_5 = (\top, \top)$	$CP_5 = (\top, \top)$

Constant Propagation Analysis with Assertions

An Example

Example 8.7

```
if [x = 1]^1 then
  [assert x = 1]^2;
  [y := x + 1]^3
else
  [assert ¬(x = 1)]^4;
  [y := 2]^5
end;
[skip]^6
```

Without assertions	With assertions
$CP_1 = (\top, \top)$	$CP_1 = (\top, \top)$
	$CP_2 = (\top, \top)$
$CP_3 = (\top, \top)$	$CP_3 = (1, \top)$
	$CP_4 = (\top, \top)$
$CP_5 = (\top, \top)$	$CP_5 = (\top, \top)$
$CP_6 = (\top, \top) \sqcup (\top, 2)$	$CP_6 = (1, 2) \sqcup (\top, 2)$
$= (\top, \top)$	$= (\top, 2)$