Static Program Analysis

Lecture 8: Dataflow Analysis VII (Narrowing & DFA with Conditional Branches)

Winter Semester 2016/17

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https://moves.rwth-aachen.de/teaching/ws-1617/spa/
Organisational Matters

Lecture Thu 01.12.2016 → Tue 06.12.2016 AH 2
Outline of Lecture 8

Revision: Undecidability of the MOP Solution

Recap: Interval Analysis

Narrowing

Taking Conditional Branches into Account

Constant Propagation Analysis with Assertions
Undecidability of the MOP Solution I

Theorem (Undecidability of MOP solution)

The MOP solution for Constant Propagation is undecidable.

Proof.

Based on undecidability of Modified Post Correspondence Problem (MPCP):
Let $\Gamma$ be some alphabet, $n \in \mathbb{N}$, and $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Gamma^+$. Do there exist $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with $m \geq 1$ and $i_1 = 1$ such that $u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$?
Theorem (Undecidability of MOP solution)

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Based on undecidability of Modified Post Correspondence Problem (MPCP):
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Given a MPCP, we construct a WHILE program (with strings and Booleans) whose MOP analysis detects a constant property iff the MPCP has no solution (see next slide).
Revision: Undecidability of the MOP Solution

Undecidability of the MOP Solution II

Proof (continued).

\[
x := u_1; y := v_1;
\]

\[\text{while} \ldots \text{do} \]
\[\quad \text{if} \ldots \text{then}
\]
\[\quad \quad x := x ++ u_1;
\]
\[\quad \quad y := y ++ v_1
\]
\[\quad \text{else if} \ldots \text{then}
\]
\[\quad \quad : \]
\[\quad \text{else}
\]
\[\quad \quad x := x ++ u_n;
\]
\[\quad \quad y := y ++ v_n
\]
\[\quad \text{end} \ldots \text{end}
\]
\[\text{end}; \]
\[\text{if} \ x = y \ \text{then} \ z := 1 \ \text{else} \ z := 0 \ \text{end};
\]
\[z := (x = y);
\]
\[\text{[skip]}\]
Proof (continued).

\[ x := u_1; y := v_1; \]
\[ \text{while } \ldots \text{ do} \]
\[ \quad \text{if } \ldots \text{ then} \]
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\[ \quad \quad \ldots \]
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\[ \quad \quad x := x ++ u_n; \]
\[ \quad \quad y := y ++ v_n \]
\[ \text{end } \ldots \text{ end} \]
\[ \text{end}; \]
\[ \text{if } x = y \text{ then } z := 1 \text{ else } z := 0 \text{ end}; \]
\[ z := (x = y); \]
\[ [\text{skip}]' \]

Then: \( \text{mop}(l)(z) = \text{false} \)
Proof (continued).

\[ x := u_1; y := v_1; \]

while ... do

\[ \text{if ... then} \]

\[ x := x ++ u_1; \]
\[ y := y ++ v_1 \]

\[ \text{else if ... then} \]

\[ : \]

\[ \text{else} \]

\[ x := x ++ u_n; \]
\[ y := y ++ v_n \]

\[ \text{end ... end} \]

end;

if \( x = y \) then \( z := 1 \) else \( z := 0 \) end;

\[ z := (x = y); \]

[skip]

Then: \( \text{mop}(l)(z) = \text{false} \)

\[ \iff \ x \neq y \text{ at the end of every path up to } l \]
Proof (continued).

\[
x := u_1; y := v_1;
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while ... do
  if ... then
    \[
x := x ++ u_1;
y := y ++ v_1
    \]
  else if ... then
    :
  else
    \[
x := x ++ u_n;
y := y ++ v_n
    \]
end ... end
end;
if \( x = y \) then \( z := 1 \) else \( z := 0 \) end;
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z := (x = y);
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Then: \( \text{mop}(l)(z) = \text{false} \)

\[\iff x \neq y \text{ at the end of every path up to } l\]

\[\iff \text{the MPCP has no solution}\]
Recap: Interval Analysis

Outline of Lecture 8

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Recap: Interval Analysis

The Complete Lattice of Interval Analysis

\[\left\{\left[\infty, +\infty\right], \emptyset, \left\{\left[-\infty, -2\right], \left[-2, -2\right]\right\}, \ldots, \left\{\left[-\infty, 0\right], \left[-2, 0\right]\right\}, \ldots, \left\{\left[-\infty, -1\right], \left[-2, -1\right]\right\}, \ldots, \left\{\left[0, +\infty\right], \left[1, +\infty\right]\right\}\right\}\]
Recap: Interval Analysis

Formalising Interval Analysis I

The dataflow system \( S = (\text{Lab}, E, F, (D, \sqsubseteq), \iota, \varphi) \) is given by

- set of labels \( \text{Lab} := \text{Lab}_c \)
- extremal labels \( E := \{\text{init}(c)\} \) (forward problem)
- flow relation \( F := \text{flow}(c) \) (forward problem)
- complete lattice \( (D, \sqsubseteq) \) where
  - \( D := \{\delta \mid \delta : \text{Var}_c \rightarrow \text{Int}\} \)
  - \( \delta_1 \sqsubseteq \delta_2 \) iff \( \delta_1(x) \subseteq \delta_2(x) \) for every \( x \in \text{Var}_c \)
- \( \iota := \top_D : \text{Var}_c \rightarrow \text{Int} : x \mapsto \top_{\text{Int}} \) (with \( \top_{\text{Int}} = [-\infty, +\infty] \))
- \( \varphi \): see next slide
Recap: Interval Analysis

Formalising Interval Analysis II

Transfer functions \( \{ \varphi_l \mid l \in \text{Lab} \} \) are defined by

\[
\varphi_l(\delta) := \begin{cases} 
\delta & \text{if } B^l = \text{skip} \text{ or } B^l \in \text{BExp} \\
\delta[x \mapsto \text{val}_\delta(a)] & \text{if } B^l = (x := a)
\end{cases}
\]

where

\[
\text{val}_\delta(x) := \delta(x) \\
\text{val}_\delta(z) := [z, z] \\
\text{val}_\delta(a_1 + a_2) := \text{val}_\delta(a_1) \oplus \text{val}_\delta(a_2) \\
\text{val}_\delta(a_1 - a_2) := \text{val}_\delta(a_1) \ominus \text{val}_\delta(a_2) \\
\text{val}_\delta(a_1 \cdot a_2) := \text{val}_\delta(a_1) \odot \text{val}_\delta(a_2)
\]

with

\[
\emptyset \oplus J := J \oplus \emptyset := \emptyset \ominus J := \ldots := \emptyset
\]

\[
[y_1, y_2] \oplus [Z_1, Z_2] := [y_1 + Z_1, y_2 + Z_2] \\
[y_1, y_2] \ominus [Z_1, Z_2] := [y_1 - Z_2, y_2 - Z_1] \\
[y_1, y_2] \odot [Z_1, Z_2] := \bigsqcup \{y_1 z_1, y_1 z_2, y_2 z_1, y_2 z_2\} \cup \bigsqcap \{y_1 z_1, y_1 z_2, y_2 z_1, y_2 z_2\}
\]
Recap: Interval Analysis

Widening Operators

Definition (Widening operator)

Let \((D, \sqsubseteq)\) be a complete lattice. A mapping \(\nabla : D \times D \to D\) is called **widening operator** if

- for every \(d_1, d_2 \in D\),
  
  \[d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2\]

  and

- for all ascending chains \(d_0 \sqsubseteq d_1 \sqsubseteq \ldots\), the ascending chain \(d_0^\nabla \sqsubseteq d_1^\nabla \sqsubseteq \ldots\) eventually stabilises where
  
  \[d_0^\nabla := d_0\text{ and } d_{i+1}^\nabla := d_i^\nabla \nabla d_{i+1}\text{ for each } i \in \mathbb{N}\]

Remarks:

- \((d_i^\nabla)_{i \in \mathbb{N}}\) is clearly an ascending chain as \(d_{i+1}^\nabla = d_i^\nabla \nabla d_{i+1} \sqsupseteq d_i^\nabla \sqcup d_{i+1} \sqsupseteq d_i^\nabla\)

- In contrast to \(\sqcup\), \(\nabla\) does not have to be commutative, associative, monotonic, nor absorptive \((d \nabla d = d)\)

- The requirement \(d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2\) guarantees **soundness** of widening
Recap: Interval Analysis

Applying Widening to Interval Analysis

- A widening operator: $\nabla : \text{Int} \times \text{Int} \rightarrow \text{Int}$ with
  
  $$
  \emptyset \nabla J := J \nabla \emptyset := J
  $$

  $$
  [x_1, x_2] \nabla [y_1, y_2] := [z_1, z_2]
  $$

  where

  $$
  z_1 := \begin{cases} x_1 & \text{if } x_1 \leq y_1 \\ -\infty & \text{otherwise} \end{cases}
  $$

  $$
  z_2 := \begin{cases} x_2 & \text{if } x_2 \geq y_2 \\ +\infty & \text{otherwise} \end{cases}
  $$

- Widening turns infinite ascending chain
  
  $$
  J_0 = \emptyset \subseteq J_1 = [1, 1] \subseteq J_2 = [1, 2] \subseteq J_3 = [1, 3] \subseteq \ldots
  $$

  into a finite one:

  $$
  J_0^\nabla = J_0 = \emptyset
  $$

  $$
  J_1^\nabla = J_0^\nabla \nabla J_1 = \emptyset \nabla [1, 1] = [1, 1]
  $$

  $$
  J_2^\nabla = J_1^\nabla \nabla J_2 = [1, 1] \nabla [1, 2] = [1, +\infty]
  $$

  $$
  J_3^\nabla = J_2^\nabla \nabla J_3 = [1, +\infty] \nabla [1, 3] = [1, +\infty]
  $$

- In fact, the maximal chain size arising with this operator is 4:

  $$
  \emptyset \subseteq [3, 7] \subseteq [3, +\infty] \subseteq [-\infty, +\infty]
  $$
Recap: Interval Analysis

Worklist Algorithm with Widening

**Goal:** extend Algorithm 5.1 by widening to ensure termination

**Algorithm (Worklist algorithm)**

- **Input:** dataflow system $S = (\text{Lab}, E, F, (D, \sqsubseteq), \iota, \varphi)$
- **Variables:** $W \in (\text{Lab} \times \text{Lab})^*$, $\{\text{AI}_l \in D \mid l \in \text{Lab}\}$
- **Procedure:**
  - $W := \varepsilon$; **for** $(l, l') \in F$ **do** $W := W \cdot (l, l')$; % Initialise W
  - **for** $l \in \text{Lab}$ **do**
    - if $l \in E$ then $\text{AI}_l := \iota$ else $\text{AI}_l := \perp_D$; % Initialise AI
  - while $W \neq \varepsilon$ do
    - $(l, l') := \text{head}(W)$; $W := \text{tail}(W)$; % Next control-flow edge
    - if $\varphi_l(\text{AI}_l) \nsubseteq \text{AI}_{l'}$ then
      - $\text{AI}_{l'} := \text{AI}_{l'} \cup \varphi_l(\text{AI}_l)$; % Update analysis information
    - **for** $(l', l'') \in F$ **do**
      - if $(l'', l''')$ not in $W$ then $W := (l', l''') \cdot W$; % Propagate modification
  - **Output:** $\{\text{AI}_l \mid l \in \text{Lab}\}$

Remark: due to widening, only fix $\nabla(S)$ with widening is guaranteed (cf. Thm. 5.4)
Recap: Interval Analysis

Worklist Algorithm with Widening

**Goal:** extend Algorithm 5.1 by widening to ensure termination

**Algorithm (Worklist algorithm with widening)**

```
Input: dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$
Variables: $W \in (Lab \times Lab)^*$, $\{AI_l \in D \mid l \in Lab\}$
Procedure: $W := \varepsilon$; for $(l, l') \in F$ do $W := W \cdot (l, l')$; % Initialise $W$
    for $l \in Lab$ do
        if $l \in E$ then $AI_l := \iota$ else $AI_l := \bot_D$; % Initialise $AI$
    while $W \neq \varepsilon$ do
        $(l, l') := \text{head}(W)$; $W := \text{tail}(W)$; % Next control-flow edge
        if $\varphi_l(AI_l) \not\sqsubseteq AI_{l'}$ then % Fixpoint not yet reached
            $AI_{l'} := AI_{l'} \triangledown \varphi_l(AI_l)$; % Update analysis information
        for $(l', l'') \in F$ do
            if $(l'', l'')$ not in $W$ then $W := (l', l'') \cdot W$; % Propagate modification
Output: $\{AI_l \mid l \in Lab\}$, denoted by $\text{fix}^\triangledown(\Phi_S)$
```
Recap: Interval Analysis

Worklist Algorithm with Widening

Goal: extend Algorithm 5.1 by widening to ensure termination

Algorithm (Worklist algorithm with widening)

- **Input:** dataflow system $S = (\text{Lab}, E, F, (D, \sqsubseteq), \iota, \varphi)$
- **Variables:** $W \in (\text{Lab} \times \text{Lab})^*$, $\{\text{Al}_l \in D \mid l \in \text{Lab}\}$
- **Procedure:**
  
  - $W := \varepsilon$; for $(l, l') \in F$ do $W := W \cdot (l, l')$; % Initialise $W$
  - for $l \in \text{Lab}$ do
    - if $l \in E$ then $\text{Al}_l := \iota$ else $\text{Al}_l := \perp_D$; % Initialise $\text{Al}$
  - while $W \neq \varepsilon$ do
    - $(l, l') := \text{head}(W)$; $W := \text{tail}(W)$; % Next control-flow edge
    - if $\varphi_l(\text{Al}_l) \nsubseteq \text{Al}_{l'}$ then $\text{Al}_{l'} := \text{Al}_{l'} \triangledown \varphi_l(\text{Al}_l)$; % Update analysis information
    - for $(l', l'') \in F$ do
      - if $(l'', l''')$ not in $W$ then $W := (l', l''') \cdot W$; % Propagate modification
  - **Output:** $\{\text{Al}_l \mid l \in \text{Lab}\}$, denoted by $\text{fix}^{\triangledown}(\Phi_S)$

Remark: due to widening, only $\text{fix}^{\triangledown}(\Phi_S) \supseteq \text{fix}(\Phi_S)$ is guaranteed (cf. Thm. 5.4)
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Another Widening Example

Example 8.1

Transfer functions (for $\delta = (J_x, J_y)$):

\[
\begin{align*}
\varphi_1(J_x, J_y) &= ([1, 1], J_y) \\
\varphi_2(J_x, J_y) &= (J_x, [2, 2]) \\
\varphi_3(J_x, J_y) &= (J_x, J_y) \\
\varphi_4(J_x, J_y) &= ([3, 3], J_y) \\
\varphi_5(J_x, \emptyset) &= (J_x, \emptyset) \\
\varphi_5(J_x, [y_1, y_2]) &= (J_x, [y_1 + 1, y_2 + 1])
\end{align*}
\]
Narrowing

Another Widening Example

Example 8.1

\[
\begin{align*}
[&x := 1] &  \\
[&y := 2] &  \\
[\text{while [...]}] &  \\
[&x := 3] &  \\
[&y := y + 1] & \\
\end{align*}
\]

Transfer functions (for \(\delta = (J_x, J_y)\)):
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\end{align*}
\]

Application of worklist algorithm

1. without widening (omitted): diverges (for \(y\)) with expected result for \(x\): \(\text{AI}_3(x) = [1, 3]\)
2. with widening (on the board): terminates with unexpected result for \(x\): \(\text{AI}_3(x) = [1, +\infty]\)
Narrowing

Idea of Narrowing

- **Observation**: widening can “shoot above the target”, i.e., lead to *unnecessarily imprecise results*
Narrowing

Idea of Narrowing

- **Observation:** widening can “shoot above the target”, i.e., lead to unnecessarily imprecise results
- **Solution:** improvement by iterating again from the result obtained by widening (i.e., from $\text{fix}^\triangledown(\Phi_S)$)

$$\implies \text{compute } \Phi^k_S(\text{fix}^\triangledown(\Phi_S)) \text{ for } k = 1, 2, \ldots$$
Narrowing

Idea of Narrowing

• **Observation:** widening can “shoot above the target”, i.e., lead to unnecessarily imprecise results

• **Solution:** improvement by iterating again from the result obtained by widening (i.e., from $\text{fix}^\wedge (\Phi_S)$)

  $$\implies \text{compute } \Phi^k_S(\text{fix}^\wedge (\Phi_S)) \text{ for } k = 1, 2, \ldots$$

• **Soundness:** $\text{fix}^\wedge (\Phi_S) \supseteq \text{fix}(\Phi_S)$ (cf. Alg. 7.5)

  $$\implies \Phi^k_S(\text{fix}^\wedge (\Phi_S)) \supseteq \Phi^k_S(\text{fix}(\Phi_S)) = \text{fix}(\Phi_S)$$

  (since $\Phi_S$ and thus $\Phi^k_S$ monotonic)
Narrowing

Example 8.2 (cf. Example 8.1)

Transfer functions (for \( \delta = (J_x, J_y) \)):

\[
\begin{align*}
\varphi_1(J_x, J_y) &= ([1, 1], J_y) \\
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<th>$\text{Al}_4$</th>
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<td>$\text{fix}^\nabla(\Phi_S)$</td>
<td>$(\top, \top)$</td>
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<tr>
<th>Narrowing</th>
<th>$\text{Al}_1$</th>
<th>$\text{Al}_2$</th>
<th>$\text{Al}_3$</th>
<th>$\text{Al}_4$</th>
<th>$\text{Al}_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{fix}^\triangledown(\Phi_S)$</td>
<td>(T, T)</td>
<td>([1, 1], T)</td>
<td>([1, +\infty], [2, +\infty])</td>
<td>([1, +\infty], [2, +\infty])</td>
<td>([3, 3], [2, +\infty])</td>
</tr>
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<td>$\Phi_S(\text{fix}^\triangledown(\Phi_S))$</td>
<td>(T, T)</td>
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<td>([1, 3], [2, +\infty])</td>
<td>([1, +\infty], [2, +\infty])</td>
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</tr>
<tr>
<td>$\Phi_S^2(\text{fix}^\triangledown(\Phi_S))$</td>
<td>(T, T)</td>
<td>([1, 1], T)</td>
<td>([1, 3], [2, +\infty])</td>
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<td>([1, 1], T)</td>
<td>([1, 3], [2, +\infty])</td>
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</tbody>
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Narrowing

Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)
Narrowing

Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)
- **But:** possible to stop after each step without losing soundness ($\Phi^k_S(\text{fix} \downarrow (\Phi_S)) \sqsubseteq \text{fix}(\Phi_S)$)
Narrowing in Practice

- **Problem:** narrowing may not terminate (due to infinite descending chains)
- **But:** possible to stop after each step without losing soundness \((\Phi^k_S(\text{fix}^\nabla(\Phi_S)) \equiv \text{fix}(\Phi_S))\)
- **In practice:** termination often ensured by using narrowing operators
  \((\simeq\) counterpart of widening operator; definition omitted)
Taking Conditional Branches into Account

Outline of Lecture 8

Revision: Undecidability of the MOP Solution

Recap: Interval Analysis

Narrowing

Taking Conditional Branches into Account

Constant Propagation Analysis with Assertions
Taking Conditional Branches into Account

Taking Conditional Branches into Account I

- **So far:** values of conditions have been ignored in analysis
- Essentially: *if* and *while* statements treated as *nondeterministic choice* between the two branches

Example 8.3

```plaintext
y := 0;
z := 0;
while [x > 0] do
  if y < 17 then
    y := y + 1
  end;
  z := z + x;
x := x - 1
end;
```

- Interval analysis (with widening) yields for l:
  - \( x \in [-\infty, +\infty] \)
  - \( y \in [0, +\infty] \)
  - \( z \in [-\infty, +\infty] \)

- Too pessimistic! In fact,
  - \( x \in [-\infty, +\infty] \)
  - \( y \in [0, 17] \)
  - \( z \in [0, +\infty] \)
Taking Conditional Branches into Account

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Taking Conditional Branches into Account

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while [x > 0]' do
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  end;
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x := x - 1
end;
```

- Interval analysis (with widening) yields for /:
  
  \[
  x \in [-\infty, +\infty] \\
  y \in [0, +\infty] \\
  z \in [-\infty, +\infty]
  \]
Taking Conditional Branches into Account

- **So far:** values of conditions have been ignored in analysis
- Essentially: *if* and *while* statements treated as **nondeterministic choice** between the two branches

**Example 8.3**

```
y := 0;
z := 0;
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```

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  - \( x \in [-\infty, +\infty] \)
  - \( y \in [0, +\infty] \)
  - \( z \in [-\infty, +\infty] \)

- **Too pessimistic!** In fact,
  - \( x \in [-\infty, +\infty] \)
  - \( y \in [0, 17] \)
  - \( z \in [0, +\infty] \)
Taking Conditional Branches into Account

Taking Conditional Branches into Account II

- **Solution:** introduce transfer functions for branches
Taking Conditional Branches into Account

Taking Conditional Branches into Account II

- **Solution**: introduce transfer functions for branches
- **First approach**: attach (negated) conditions as labels to control flow edges
  - advantage: no language modification required
  - disadvantage: entails extension of DFA framework
  - will not further be considered here
Taking Conditional Branches into Account

Taking Conditional Branches into Account II

- **Solution:** introduce transfer functions for branches
- **First approach:** attach (negated) conditions as labels to control flow edges
  - advantage: no language modification required
  - disadvantage: entails extension of DFA framework
  - will not further be considered here
- **Second approach:** encode conditions as assertions (proper statements)
  - advantage: DFA framework can be reused
  - disadvantage: entails extension of WHILE language
  - the way we will follow
Taking Conditional Branches into Account

Conditions as Edge Labels

**Example 8.4 (cf. Example 8.3)**

<table>
<thead>
<tr>
<th>Step</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ y := 0 ]</td>
</tr>
<tr>
<td>2</td>
<td>[ z := 0 ]</td>
</tr>
<tr>
<td>3</td>
<td>while [ x &gt; 0 ]</td>
</tr>
<tr>
<td>4</td>
<td>if [ y &lt; 17 ]</td>
</tr>
<tr>
<td>5</td>
<td>[ y := y + 1 ]</td>
</tr>
<tr>
<td>6</td>
<td>[ z := z + x ]</td>
</tr>
<tr>
<td>7</td>
<td>[ x := x - 1 ]</td>
</tr>
</tbody>
</table>

Diagram:
- **Step 1:** \[ y := 0 \]
- **Step 2:** \[ z := 0 \]
- **Step 3:** while \[ x > 0 \]
- **Step 4:** if \[ y < 17 \]
- **Step 5:** \[ y := y + 1 \]
- **Step 6:** \[ z := z + x \]
- **Step 7:** \[ x := x - 1 \]

Conditions as Edge Labels:
- \[ x > 0 \]
- \[ y < 17 \]
- \[ y < 17 \] (negated)
- \[ x > 0 \] (negated)
Taking Conditional Branches into Account

Conditions as Edge Labels vs. Conditions as Assertions

Example 8.4 (cf. Example 8.3)

\[
\begin{align*}
\text{y := 0}^1 \\
\text{z := 0}^2 \\
\text{while x > 0}^3 \\
\text{if y < 17}^4 \\
\text{y := y + 1}^5 \\
\text{z := z + x}^6 \\
\text{x := x - 1}^7 \\
\neg(x > 0) \\
\neg(y < 17)
\end{align*}
\]

\[
\begin{align*}
y := 0; \\
z := 0; \\
\text{while x > 0 do} \\
\quad \text{assert x > 0;} \\
\quad \text{if y < 17 then} \\
\quad \quad \text{assert y < 17;} \\
\quad \quad y := y + 1 \\
\quad \quad \text{end;} \\
\quad z := z + x; \\
\quad x := x - 1 \\
\quad \text{end;} \\
\quad \text{assert } \neg(x > 0);
\end{align*}
\]
Taking Conditional Branches into Account

Extending the Syntax of WHILE Programs

**Definition 8.5 (Labelled WHILE programs with assertions)**

The syntax of labelled WHILE programs with assertions is defined by the following context-free grammar:

\[
\begin{align*}
a & ::= z \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \cdot a_2 \in AExp \\
b & ::= t \mid a_1 = a_2 \mid a_1 > a_2 \mid \neg b \mid b_1 \land b_2 \mid b_1 \lor b_2 \in BExp \\
c & ::= [\text{skip}]' \mid [x := a]' \mid c_1 ; c_2 \\
    \quad \text{if } [b]' \text{ then } c_1 \text{ else } c_2 \text{ end} \mid \text{while } [b]' \text{ do } c \text{ end} \mid [\text{assert } b]' \in Cmd
\end{align*}
\]
Taking Conditional Branches into Account

Extending the Syntax of WHILE Programs

Definition 8.5 (Labelled WHILE programs with assertions)

The syntax of labelled WHILE programs with assertions is defined by the following context-free grammar:

\[
\begin{align*}
a & ::= z | x | a_1 + a_2 | a_1 - a_2 | a_1 * a_2 \in AExp \\
b & ::= t | a_1 = a_2 | a_1 > a_2 | \neg b | b_1 \land b_2 | b_1 \lor b_2 \in BExp \\
c & ::= [\text{skip}]' | [x := a]' | c_1; c_2 \\
& \quad \text{if} [b]' \text{ then } c_1 \text{ else } c_2 \text{ end} | \text{while} [b]' \text{ do } c \text{ end} | [\text{assert } b]' \in Cmd
\end{align*}
\]

To be done:

- Definition of transfer functions for assert blocks (depending on analysis problem)
- Idea: assertions as filters that let only “compatible” information pass
Constant Propagation Analysis with Assertions

Outline of Lecture 8

Revision: Undecidability of the MOP Solution

Recap: Interval Analysis

Narrowing

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Constant Propagation Analysis with Assertions
Constant Propagation Analysis with Assertions

Original Constant Propagation Analysis

So far:

- complete lattice \((D, \sqsubseteq)\) where
  - \(D := \{ \delta \mid \delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}\}\)
    - \(\delta(x) = z \in \mathbb{Z}: x \text{ has constant value } z\) (i.e., possible values in \(\{z\}\))
    - \(\delta(x) = \bot: x \text{ undefined }\) (i.e., possible values in \(\emptyset\))
    - \(\delta(x) = \top: x \text{ overdefined}\) (i.e., possible values in \(\mathbb{Z}\))
  - \(\sqsubseteq \subseteq D \times D\) defined by pointwise extension of \(\bot \subseteq z \subseteq \top\) (for every \(z \in \mathbb{Z}\))
- transfer functions \(\{ \varphi_l \mid l \in \text{Lab} \}\) defined by
  \[
  \varphi_l(\delta) := \begin{cases} 
  \delta & \text{if } B^l = \text{skip} \text{ or } B^l \in \text{BExp} \\
  \delta[x \mapsto \text{val}_\delta(a)] & \text{if } B^l = (x := a)
  \end{cases}
  \]

  where

  \[
  \text{val}_\delta(x) := \delta(x) \quad \text{val}_\delta(z) := z \quad \text{val}_\delta(a_1 \text{ op } a_2) := \begin{cases} 
  z_1 \text{ op } z_2 & \text{if } z_1, z_2 \in \mathbb{Z} \\
  \bot & \text{if } z_1 = \bot \text{ or } z_2 = \bot \\
  \top & \text{otherwise}
  \end{cases}
  \]

  for \(z_1 := \text{val}_\delta(a_1)\) and \(z_2 := \text{val}_\delta(a_2)\)
Constant Propagation Analysis with Assertions

Transfer Functions of Assertions I

Additionally for $B^l = (\text{assert } b)$, $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$ and $x \in \text{Var}_c$:

$$\varphi_l(\delta)(x) := \begin{cases} 
\bot & \text{if } \not\exists \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \\
z & \text{if } \forall \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \implies \sigma(x) = z \\
\top & \text{otherwise}
\end{cases}$$
Constant Propagation Analysis with Assertions

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Additionally for $B^l = (\text{assert } b)$, $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$ and $x \in \text{Var}_c$:

$$\varphi_l(\delta)(x) := \begin{cases} 
\bot & \text{if } \nexists \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \\
z & \text{if } \forall \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \implies \sigma(x) = z \\
\top & \text{otherwise}
\end{cases}$$

where

- the set of $\delta$-states is given by

$$\Sigma_\delta := \left\{ \sigma : \text{Var}_c \rightarrow \mathbb{Z} \mid \forall y \in \text{Var}_c : \sigma(y) \in \begin{cases} 
\emptyset & \text{if } \delta(y) = \bot \\
\{z\} & \text{if } \delta(y) = z \\
\mathbb{Z} & \text{if } \delta(y) = \top
\end{cases} \right\}$$

(and thus $\Sigma_\delta = \emptyset$ iff $\delta(y) = \bot$ for some $y \in \text{Var}_c$)
Constant Propagation Analysis with Assertions

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Additionally for $B^l = (\text{assert } b)$, $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$ and $x \in \text{Var}_c$:

$$\varphi_l(\delta)(x) := \begin{cases} 
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\end{cases} \right\}$$

(and thus $\Sigma_{\delta} = \emptyset$ iff $\delta(y) = \bot$ for some $y \in \text{Var}_c$)

- the evaluation function $\text{val}_\sigma : B\text{Exp} \rightarrow \mathbb{B}$ for $\sigma : \text{Var}_c \rightarrow \mathbb{Z}$ is defined by

  - $\text{val}_\sigma(t) := t$
  - $\text{val}_\sigma(\neg b) := \begin{cases} 
\text{true} & \text{if } \text{val}_\sigma(b) = \text{false} \\
\text{false} & \text{otherwise}
\end{cases}$
  - $\text{val}_\sigma(a_1 = a_2) := (\text{val}_\sigma(a_1) = \text{val}_\sigma(a_2))$
  - $\text{val}_\sigma(b_1 \land b_2) := \begin{cases} 
\text{true} & \text{if } \text{val}_\sigma(b_1) = \text{val}_\sigma(b_2) = \text{true} \\
\text{false} & \text{otherwise}
\end{cases}$
Example 8.6

1. \(\text{Var}_c = \{x, y, z\}, \delta = (\bot_x, 1_y, \top_z)\)
\[\implies \Sigma_\delta = \emptyset \implies \varphi_{\text{assert } b(\delta)} = (\bot, \bot, \bot)\text{ for every } b \in B\text{Exp}\]
Transfer Functions of Assertions II

Example 8.6

1. \( \text{Var}_c = \{x, y, z\}, \delta = (\bot_x, 1_y, \top_z) \)

   \( \implies \Sigma_\delta = \emptyset \implies \varphi_{\text{assert }} b(\delta) = (\bot, \bot, \bot) \) for every \( b \in \text{BExp} \)

2. \( \text{Var}_c = \{x, y, z\}, \delta = (1_x, 2_y, \top_z) \)

   \( \implies \Sigma_\delta = \{(1, 2, z) \mid z \in \mathbb{Z}\} \implies \varphi_{\text{assert }} x=y(\delta) = (\bot, \bot, \bot) \)
   \( \varphi_{\text{assert }} y=z(\delta) = (1, 2, 2) \)
   \( \varphi_{\text{assert }} y<z(\delta) = (1, 2, \top) \)
   \( \varphi_{\text{assert }} x<=z \land y>z(\delta) = (1, 2, 1) \)
Constant Propagation Analysis with Assertions

Transfer Functions of Assertions II

Example 8.6

1. \( \text{Var}_c = \{x, y, z\}, \ \delta = (\bot_x, 1_y, \top_z) \)
   \[ \Rightarrow \ \Sigma_\delta = \emptyset \Rightarrow \varphi_{\text{assert}} b(\delta) = (\bot, \bot, \bot) \text{ for every } b \in B\text{Exp} \]
2. \( \text{Var}_c = \{x, y, z\}, \ \delta = (1_x, 2_y, 1_z) \)
   \[ \Rightarrow \ \Sigma_\delta = \{(1, 2, z) \mid z \in \mathbb{Z}\} \Rightarrow \varphi_{\text{assert}} x\neq y(\delta) = (1, 1, \bot) \]
   \[ \varphi_{\text{assert}} y\neq z(\delta) = (1, 2, 2) \]
   \[ \varphi_{\text{assert}} y< z(\delta) = (1, 2, \top) \]
   \[ \varphi_{\text{assert}} \text{x}<= \text{z} \wedge \text{y} > \text{z}(\delta) = (1, 2, 1) \]
3. \( \text{Var}_c = \{x, y, z\}, \ \delta = (1_x, \top_y, \top_z) \)
   \[ \Rightarrow \ \Sigma_\delta = \{(1, z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}\} \Rightarrow \varphi_{\text{assert}} x\neq y(\delta) = (1, 1, \top) \]
   \[ \varphi_{\text{assert}} y\neq z(\delta) = (1, \top, \top) \]
Transfer Functions of Assertions III

Remarks:
- For $B' = (\text{assert } b)$ and $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
Transfer Functions of Assertions III

Remarks:

- For $B' = (\text{assert } b)$ and $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}$, $\varphi_l(\delta) \subseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
- Constant propagation captures interdependencies between variables only when both are constant (cf. "assert y=z" in Example 8.6(3))
Transfer Functions of Assertions III

Remarks:

- For $B' = (\text{assert } b)$ and $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \subseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ (“filter”)
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- $\varphi_l(\delta)$ can be determined (or at least approximated) by Satisfiability Modulo Theories (SMT) techniques
Constant Propagation Analysis with Assertions

Transfer Functions of Assertions III

Remarks:

- For $B' = (\text{assert } b)$ and $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\bot, \top\}$, $\varphi_l(\delta) \subseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ ("filter")
- Constant propagation captures interdependencies between variables only when both are constant (cf. "assert $y=z$" in Example 8.6(3))
- $\varphi_l(\delta)$ can be determined (or at least approximated) by Satisfiability Modulo Theories (SMT) techniques
- If $\text{CP}_l(x) = \bot$ for some $l \in \text{Lab}_c$ and $x \in \text{Var}_c$, then $l$ is unreachable (and $\text{CP}_l(y) = \bot$ for all $y \in \text{Var}_c$)
Constant Propagation Analysis with Assertions

An Example

Example 8.7

```plaintext
if [x = 1]\(^1\) then
  [assert x = 1]\(^2\);
  [y := x + 1]\(^3\)
else
  [assert \neg(x = 1)]\(^4\);
  [y := 2]\(^5\)
end;
[skip]\(^6\)
```
Constant Propagation Analysis with Assertions

An Example

Example 8.7

\[
\begin{align*}
\text{if } [x = 1] & \quad \text{then} \\
\text{[assert } x = 1] & \quad \text{2;} \\
[y := x + 1] & \quad \text{3} \\
\text{else} & \quad \text{4} \\
\text{[assert } \neg (x = 1)] & \quad \text{4;} \\
[y := 2] & \quad \text{5} \\
\text{end;} & \quad \text{6} \\
\text{[skip]} & \quad \text{6}
\end{align*}
\]

<table>
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<tbody>
<tr>
<td>( CP_1 = (\top, \top) )</td>
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Constant Propagation Analysis with Assertions

An Example

Example 8.7

if \([x = 1]\) then
  \[\text{assert } x = 1\];
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  \[y := 2\]
end;
\[\text{skip}\]

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### Constant Propagation Analysis with Assertions

#### An Example

**Example 8.7**

```
if [x = 1]\(^1\) then
  [assert x = 1]\(^2\);
  [y := x + 1]\(^3\)
else
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  [y := 2]\(^5\)
end;
[skip]\(^6\)
```

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**Example 8.7**

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<td>$[\text{assert } x = 1]$;</td>
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<td>$[y := x + 1]$</td>
</tr>
<tr>
<td>Else</td>
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<tr>
<td>$[\text{assert } \neg (x = 1)]$;</td>
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<tr>
<td>$[y := 2]$</td>
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## Constant Propagation Analysis with Assertions

### An Example

#### Example 8.7

```plaintext
if [x = 1] then
    [assert x = 1];
    [y := x + 1]
else
    [assert ¬(x = 1)];
    [y := 2]
end;
[skip]
```

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<tr>
<td>CP₅ = (T, T)</td>
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</tbody>
</table>
```

Without assertions

With assertions
Constant Propagation Analysis with Assertions

An Example

Example 8.7

```plaintext
if [x = 1]¹ then
  [assert x = 1]²;
  [y := x + 1]³
else
  [assert ¬(x = 1)]⁴;
  [y := 2]⁵
end;
[skip]⁶
```

<table>
<thead>
<tr>
<th>Without assertions</th>
<th>With assertions</th>
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</thead>
<tbody>
<tr>
<td>CP₁ = (T, T)</td>
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<td>CP₆ = (T, T) ⊔ (T, 2) = (T, T)</td>
<td>CP₆ = (1, 2) ⊔ (T, 2) = (T, 2)</td>
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