Static Program Analysis

Lecture 3: Dataflow Analysis II (Order-Theoretic Foundations)

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https://moves.rwth-aachen.de/teaching/ws-1617/spa/
Recap: Dataflow Analysis

Outline of Lecture 3

Recap: Dataflow Analysis

Heading for a Dataflow Analysis Framework

Order-Theoretic Foundations: The Domain
Recap: Dataflow Analysis

Labelled Programs

- Goal: localisation of analysis information
- Dataflow information will be associated with
  - skip statements
  - assignments
  - tests in conditionals (if) and loops (while)
- Assume set of labels $Lab$ with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

- $a ::= z \mid x \mid a_1+a_2 \mid a_1-a_2 \mid a_1*a_2 \in AExp$
- $b ::= t \mid a_1=a_2 \mid a_1>a_2 \mid \neg b \mid b_1\land b_2 \mid b_1\lor b_2 \in BExp$
- $c ::= [\text{skip}]' \mid [x := a]' \mid c_1; c_2 \mid$
  - if $[b]'$ then $c_1$ else $c_2$ end \mid while $[b]'$ do $c$ end $\in Cmd$
- All labels in $c \in Cmd$ assumed distinct, denoted by $Lab_c$
- Labelled fragments of $c$ called blocks, denoted by $Blk_c$
Recap: Dataflow Analysis

Representing Control Flow

Example

\[ c = [z := 1] \]
\[
\text{while } [x > 0] \text{ do }
\]
\[
[z := z \cdot y] ;
[x := x - 1]
\]
end

\[ \text{init}(c) = 1 \]
\[ \text{final}(c) = \{ 2 \} \]
\[ \text{flow}(c) = \{ (1, 2), (2, 3), (3, 4), (4, 2) \} \]

Visualisation by (control) flow graph:

1. \[ z := 1 \]
2. \[ x > 0 \]
3. \[ z := z \cdot y \]
4. \[ x := x - 1 \]
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Goal of Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions must have been computed, and not later modified, on all paths to the program point.

- Can be used for Common Subexpression Elimination: replace subexpression by variable that contains up-to-date value
- Only interesting for non-trivial (i.e., complex) arithmetic expressions
Recap: Dataflow Analysis

The Equation System

• Analysis itself defined by setting up an equation system
• For each \( l \in Lab_c \), \( AE_l \subseteq CExp_c \) represents the set of available expressions at the entry of block \( B^l \)
• Formally, for \( c \in Cmd \) with isolated entry:

\[
AE_l = \begin{cases} 
\emptyset & \text{if } l = \text{init}(c) \\
\bigcap \{ \varphi_{l'}(AE_{l'}) \mid (l', l) \in \text{flow}(c) \} & \text{otherwise}
\end{cases}
\]

where \( \varphi_{l'} : 2^{CExp_c} \rightarrow 2^{CExp_c} \) denotes the transfer function of block \( B^{l'} \), given by

\[
\varphi_{l'}(A) := (A \setminus \text{kill}_{AE}(B^{l'})) \cup \text{gen}_{AE}(B^{l'})
\]

• Characterisation of analysis:
  - flow-sensitive: results depending on order of assignments
  - forward: starts in \( \text{init}(c) \) and proceeds downwards
    - must: \( \bigcap \) in equations for \( AE_l \)
• Later: solution not necessarily unique
  \( \implies \) choose greatest unique
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Goal of Live Variables Analysis

Live Variables Analysis

The goal of Live Variables Analysis is to determine, for each program point, which variables may be live at the exit from the point.

- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination: remove assignments to non-live variables
Recap: Dataflow Analysis

The Equation System

- For each $l \in Lab_c$, $LV_l \subseteq Var_c$ represents the set of live variables at the exit of block $B^l$
- Formally, for a program $c \in Cmd$ with isolated exits:
  \[
  LV_l = \begin{cases} 
  Var_c & \text{if } l \in \text{final}(c) \\
  \bigcup \{\varphi_{l'}(LV_{l'}) \mid (l, l') \in \text{flow}(c)\} & \text{otherwise}
  \end{cases}
  \]
  where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by
  \[
  \varphi_{l'}(V) := (V \setminus \text{kill}_{LV}(B^{l'})) \cup \text{gen}_{LV}(B^{l'})
  \]
- Characterisation of analysis:
  - flow-sensitive: results depending on order of assignments
  - backward: starts in $\text{final}(c)$ and proceeds upwards
    - may: $\bigcup$ in equations for $LV_l$
- Later: solution not necessarily unique
  $\implies$ choose least one
Outline of Lecture 3

Recap: Dataflow Analysis

Heading for a Dataflow Analysis Framework

Order-Theoretic Foundations: The Domain
Heading for a Dataflow Analysis Framework

Similarities Between Analysis Problems

- **Observation:** the analyses presented so far have some similarities
Heading for a Dataflow Analysis Framework

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  ⇒ Look for underlying framework
Heading for a Dataflow Analysis Framework

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- **Observation:** the analyses presented so far have some *similarities*

  ⇒ Look for underlying *framework*

- **Advantages:**
  - possibility for designing (efficient) *generic algorithms* for solving dataflow equations
  - enables generic *correctness proofs* of analyses and algorithms
Heading for a Dataflow Analysis Framework

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- **Observation:** the analyses presented so far have some similarities
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- **Advantages:**
  - possibility for designing (efficient) **generic algorithms** for solving dataflow equations
  - enables generic **correctness proofs** of analyses and algorithms

- **Overall pattern:** for \( c \in \text{Cmd} \) and \( l \in \text{Lab}_c \), the **analysis information** \((\text{AI})\) is described by **equations** of the form

\[
\text{AI}_l = \begin{cases} 
\iota & \text{if } l \in E \\
\bigsqcup \{\varphi_R(\text{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise}
\end{cases}
\]

where
- the set of extremal labels, \( E \), is \( \{\text{init}(c)\} \) or \( \text{final}(c) \)
- \( \iota \) specifies the extremal analysis information
- the combination operator, \( \bigsqcup \), is \( \cap \) or \( \cup \)
- \( \varphi_R \) denotes the **transfer function** of block \( B'' \)
- the **flow relation** \( F \) is \( \text{flow}(c) \) or \( \text{flow}^R(c) \) \( (:= \{ (l', l) \mid (l, l') \in \text{flow}(c) \}) \)
# Characterisation of Analyses

### Direction of information flow

- **Forward:**
  - $F = \text{flow}(c)$
  - $\text{AI}_I$ refers to entry of $B^I$
  - $c$ has isolated entry

- **Backward:**
  - $F = \text{flow}^R(c)$
  - $\text{AI}_I$ refers to exit of $B^I$
  - $c$ has isolated exits
# Heading for a Dataflow Analysis Framework

## Characterisation of Analyses

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  - $c$ has isolated exits

### Quantification over paths

- **May:**
  - $\bigcup = \bigcup$
  - property satisfied by some path
  - interested in least solution (later)
- **Must:**
  - $\bigcap = \bigcap$
  - property satisfied by all paths
  - interested in greatest solution (later)
Heading for a Dataflow Analysis Framework

Roadmap

**Goal:** solve dataflow equation system by *fixpoint iteration*

1. Characterise solution of equation system as *fixpoint* of a transformation
Roadmap

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1. Characterise solution of equation system as \textit{fixpoint} of a transformation
2. Introduce partial order for comparing analysis results
Heading for a Dataflow Analysis Framework

Roadmap

**Goal:** solve dataflow equation system by **fixpoint iteration**

1. Characterise solution of equation system as **fixpoint** of a transformation
2. Introduce **partial order** for comparing analysis results
3. Establish **least upper bound** as combination operator
Heading for a Dataflow Analysis Framework

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4. Ensure monotonicity of transfer functions
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**Goal:** solve dataflow equation system by **fixpoint iteration**

1. Characterise solution of equation system as **fixpoint** of a transformation
2. Introduce **partial order** for comparing analysis results
3. Establish **least upper bound** as combination operator
4. Ensure **monotonicity** of transfer functions
5. Guarantee termination of fixpoint iteration by **ascending chain condition**
Goal: solve dataflow equation system by fixpoint iteration

1. Characterise solution of equation system as fixpoint of a transformation
2. Introduce partial order for comparing analysis results
3. Establish least upper bound as combination operator
4. Ensure monotonicity of transfer functions
5. Guarantee termination of fixpoint iteration by ascending chain condition
6. Optimise fixpoint iteration by worklist algorithm
Motivation

- **Wanted**: solution of (dataflow) equation system

\[
\begin{align*}
\text{Wanted: } & \text{solution of (dataflow) equation system} \\
\end{align*}
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- **Problem:** recursive dependencies between dataflow variables
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- **Problem:** recursive dependencies between dataflow variables
- **Idea:** characterise solution as fixpoint of transformation:

\[(\text{Al}_l = \tau_l)_{l \in \text{Lab}_c} \iff \Phi((\text{Al}_l)_{l \in \text{Lab}_c}) = (\text{Al}_l)_{l \in \text{Lab}_c}\]

where \(\Phi((\text{Al}_l)_{l \in \text{Lab}_c}) := (\tau_l)_{l \in \text{Lab}_c}\)

**Approach:** approximate fixpoint by iteration
Motivation

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- **Problem:** recursive dependencies between dataflow variables
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- **Approach:** approximate fixpoint by iteration
Order-Theoretic Foundations: The Domain

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Order-Theoretic Foundations: The Domain
Partial Orders

The domain of analysis information usually forms a partial order where the ordering relation compares the “precision” of information.

Definition 3.1 (Partial order)

A partial order (PO) \((D, \sqsubseteq)\) consists of a set \(D\), called domain, and of a relation \(\sqsubseteq \subseteq D \times D\) such that, for every \(d_1, d_2, d_3 \in D\),

reflexivity: \(d_1 \sqsubseteq d_1\)

transitivity: \(d_1 \sqsubseteq d_2\) and \(d_2 \sqsubseteq d_3\) \(\implies\) \(d_1 \sqsubseteq d_3\)

antisymmetry: \(d_1 \sqsubseteq d_2\) and \(d_2 \sqsubseteq d_1\) \(\implies\) \(d_1 = d_2\)

It is called total if, in addition, always \(d_1 \sqsubseteq d_2\) or \(d_2 \sqsubseteq d_1\).
Order-Theoretic Foundations: The Domain

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**Example 3.2**

1. \((\mathbb{N}, \leq)\) is a total partial order
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# Order-Theoretic Foundations: The Domain

## Upper Bounds

In the dataflow equation system, analysis information from several predecessors is combined by taking the least upper bound.

### Definition 3.3 ((Least) upper bound)

Let \( (D, \sqsubseteq) \) be a partial order and \( S \subseteq D \).

1. An element \( d \in D \) is called an **upper bound** of \( S \) if \( s \sqsubseteq d \) for every \( s \in S \) (notation: \( S \sqsubseteq d \)).
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**Example 3.4**

1. \(S \subseteq \mathbb{N}\) has a LUB in \((\mathbb{N}, \leq)\) iff it is finite.
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1. \(S \subseteq \mathbb{N}\) has a LUB in \((\mathbb{N}, \leq)\) iff it is finite
2. (Live Variables) \((D, \sqsubseteq) = (2^{\text{Var}_c}, \subseteq)\). Given \(V_1, \ldots, V_n \subseteq \text{Var}_c\),
   \[
   \bigsqcup\{V_1, \ldots, V_n\} = \bigcup\{V_1, \ldots, V_n\}
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3. (Available Expressions) \((D, \sqsubseteq) = (2^{CExp_c}, \supseteq)\). Given \(A_1, \ldots, A_n \subseteq CExp_c\),
   \[\bigsqcup\{A_1, \ldots, A_n\} = \bigcap\{A_1, \ldots, A_n\}\]
Complete Lattices

Since \( \{ \varphi_l(Al_l) \mid (l', l) \in F \} \) can contain arbitrary elements, the existence of least upper bounds must be ensured for arbitrary subsets.

**Definition 3.5 (Complete lattice)**

A **complete lattice** is a partial order \((D, \sqsubseteq)\) such that all subsets of \(D\) have least upper bounds. In this case,

\[
\bot := \bigcup \emptyset
\]

denotes the **least element** of \(D\).
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**Example 3.6**

1. \((\mathbb{N}, \leq)\) is not a complete lattice as, e.g., \(\mathbb{N}\) does not have a LUB
Order-Theoretic Foundations: The Domain

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1. \( (\mathbb{N}, \leq) \) is not a complete lattice as, e.g., \( \mathbb{N} \) does not have a LUB
2. (Live Variables) \( (D, \subseteq) = (2^{Var_c}, \subseteq) \) is a complete lattice with \( \bot = \emptyset \)
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2. (Live Variables) \((D, \sqsubseteq) = (2^{\mathit{Var}_c}, \subseteq)\) is a complete lattice with \(\bot = \emptyset\)
3. (Available Expressions) \((D, \sqsubseteq) = (2^{\mathit{Exp}_c}, \supseteq)\) is a complete lattice with \(\bot = \mathit{Exp}_c\)
Duality in Complete Lattices

- **Dual** concept of least upper bound: greatest lower bound

- **Definitions:**
  - An element $d \in D$ is called a lower bound of $S \subseteq D$ if $d \sqsubseteq s$ for every $s \in S$ (notation: $d \sqsubseteq S$).
  - A lower bound $d$ is called greatest lower bound (GLB) or infimum of $S$ if $d' \sqsubseteq d$ for every lower bound $d'$ of $S$ (notation: $d = \bigcap S$).
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- **Examples:**
  - (Live Variables) \((D, \sqsubseteq) = (2^{Var_c}, \subseteq), \bigcap \{V_1, \ldots, V_n\} = \bigcap \{V_1, \ldots, V_n\}\)
  - (Available Expressions) \((D, \sqsubseteq) = (2^{CExp_c}, \supseteq), \bigcap \{A_1, \ldots, A_n\} = \bigcup \{A_1, \ldots, A_n\}\)

• **Lemma:** the following are equivalent:
  - \((D, \sqsubseteq)\) is a complete lattice (i.e., every subset of \( D \) has a least upper bound)
  - Every subset of \( D \) has a greatest lower bound

• **Corollary:** every complete lattice has a greatest element \( \top := \bigvee \emptyset \).
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  - (Live Variables) $(D, \sqsubseteq) = (2^{\text{Var}_c}, \subseteq), \bigcap\{V_1, \ldots, V_n\} = \bigwedge\{V_1, \ldots, V_n\}$
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- **Examples:**
  - (Live Variables) \((D, \sqsubseteq) = (2^{Var_c}, \subseteq), \bigsqcap\{V_1, \ldots, V_n\} = \cap\{V_1, \ldots, V_n\}\)
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- **Lemma:** the following are equivalent:
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Chains

Chains are generated by the approximation of the analysis information in the fixpoint iteration.

**Definition 3.7 (Chain)**

Let $(D, \sqsubseteq)$ be a partial order.

- A subset $S \subseteq D$ is called a chain in $D$ if, for every $d_1, d_2 \in S$,
  $$d_1 \sqsubseteq d_2 \text{ or } d_2 \sqsubseteq d_1$$

  (that is, $S$ is a totally ordered subset of $D$).
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2. $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots\}$ is a chain in $(2^\mathbb{N}, \subseteq)$
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The Ascending Chain Condition I

Termination of fixpoint iteration is guaranteed by the following condition.

**Definition 3.9 (Ascending Chain Condition)**

- A sequence \((d_i)_{i \in \mathbb{N}}\) is called an **ascending chain** in \(D\) if \(d_i \sqsubseteq d_{i+1}\) for each \(i \in \mathbb{N}\).
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Order-Theoretic Foundations: The Domain

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Notes:

- The finite height property implies ACC, but not vice versa (as there might be non-stabilising descending chains – see next slide)
- The complete lattice and ACC properties are orthogonal (see next slide)
Order-Theoretic Foundations: The Domain

The Ascending Chain Condition II

Example 3.10

1. \( (\mathbb{N}, \leq) \) does not satisfy ACC and is of infinite height (and not a complete lattice)
Order-Theoretic Foundations: The Domain

The Ascending Chain Condition II

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5. (Live Variables) \((2^{\text{Var}_c}, \subseteq)\) is a complete lattice satisfying ACC and is of finite height (since \(\text{Var}_c\) [unlike \(\text{Var}\)] is finite)
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Domain requirements for dataflow analysis

\((D, \sqsubseteq)\) must be a complete lattice satisfying ACC