Static Program Analysis

Lecture 3: Dataflow Analysis II (Order-Theoretic Foundations)

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https://moves.rwth-aachen.de/teaching/ws-1617/spa/

Outline of Lecture 3

Recap: Dataflow Analysis

Heading for a Dataflow Analysis Framework

Order-Theoretic Foundations: The Domain

Labelled Programs

- Goal: localisation of analysis information
- Dataflow information will be associated with
  - `skip` statements
  - assignments
  - tests in conditionals (`if`) and loops (`while`)
- Assume set of labels *Lab* with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in AExp\\
b &::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp\\
c &::= [\texttt{skip}]^l \mid [x := a]^l \mid c_1 ; c_2 \mid\\
&\quad\ \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end} \mid \texttt{while } [b]^l \texttt{ do } c \texttt{ end} \in Cmd
\end{aligned}
$$

- All labels in $c \in Cmd$ assumed distinct, denoted by Lab_c
- Labelled fragments of c called blocks, denoted by Blk_c

Recap: Dataflow Analysis

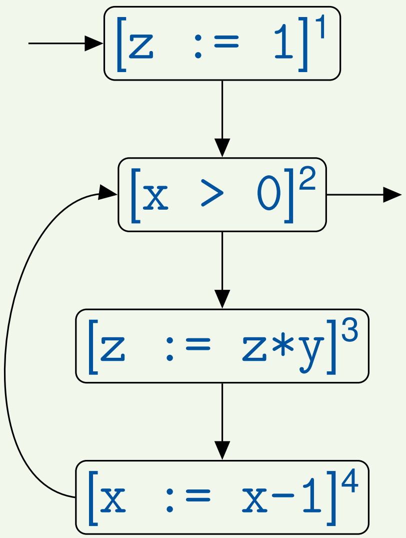
Representing Control Flow

Example

$c = [\texttt{z := 1}]^1;$
$\quad \texttt{while } [\texttt{x > 0}]^2 \texttt{ do}$
$\qquad [\texttt{z := z*y}]^3;$
$\qquad [\texttt{x := x-1}]^4$
$\quad \texttt{end}$

$$\begin{aligned} \mathrm{init}(c) &= 1 \\ \mathrm{final}(c) &= \{2\} \\ \mathrm{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\} \end{aligned}$$

Visualisation by (control) flow graph:



Goal of Available Expressions Analysis

Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

- Can be used for Common Subexpression Elimination: replace subexpression by variable that contains up-to-date value
- Only interesting for non-trivial (i.e., complex) arithmetic expressions

Recap: Dataflow Analysis

The Equation System

- Analysis itself defined by setting up an equation system
- For each $l \in Lab_c$, $\mathrm{AE}_l \subseteq CExp_c$ represents the set of available expressions at the entry of block B^l
- Formally, for $c \in Cmd$ with isolated entry:

$$\mathrm{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathrm{init}(c) \\ \bigcap\{\varphi_{l'}(\mathrm{AE}_{l'}) \mid (l', l) \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

  where $\varphi_{l'} : 2^{CExp_c} \to 2^{CExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathrm{kill}_{\mathrm{AE}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{AE}}(B^{l'})$$

- Characterisation of analysis:

  flow-sensitive: results depending on order of assignments

  forward: starts in $\mathrm{init}(c)$ and proceeds downwards

  must: $\bigcap$ in equations for AE_l
- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

Goal of Live Variables Analysis

Live Variables Analysis

The goal of Live Variables Analysis is to determine, for each program point, which variables *may* be live at the exit from the point.

- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination: remove assignments to non-live variables

Recap: Dataflow Analysis

The Equation System

- For each $l \in Lab_c$, $\mathrm{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l
- Formally, for a program $c \in Cmd$ with isolated exits:
$$\mathrm{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathrm{final}(c) \\ \bigcup\{\varphi_{l'}(\mathrm{LV}_{l'}) \mid (l, l') \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$
where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by
$$\varphi_{l'}(V) := (V \setminus \mathrm{kill}_{\mathrm{LV}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{LV}}(B^{l'})$$
- Characterisation of analysis:
  flow-sensitive: results depending on order of assignments
  backward: starts in $\mathrm{final}(c)$ and proceeds upwards
  may: $\bigcup$ in equations for LV_l
- Later: solution not necessarily unique
  $\Longrightarrow$ choose least one

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Heading for a Dataflow Analysis Framework

Order-Theoretic Foundations: The Domain

Similarities Between Analysis Problems

- **Observation:** the analyses presented so far have some similarities

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- **Advantages:**
  - possibility for designing (efficient) generic algorithms for solving dataflow equations
  - enables generic correctness proofs of analyses and algorithms

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- **Advantages:**
  - possibility for designing (efficient) generic algorithms for solving dataflow equations
  - enables generic correctness proofs of analyses and algorithms
- **Overall pattern:** for $c \in Cmd$ and $l \in Lab_c$, the analysis information (AI) is described by equations of the form

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

  where
  - the set of extremal labels, E, is $\{\mathsf{init}(c)\}$ or $\mathsf{final}(c)$
  - ι specifies the extremal analysis information
  - the combination operator, $\bigsqcup$, is $\bigcap$ or $\bigcup$
  - $\varphi_{l'}$ denotes the transfer function of block $B^{l'}$
  - the flow relation F is $\mathsf{flow}(c)$ or $\mathsf{flow}^R(c)$ $(:= \{(l', l) \mid (l, l') \in \mathsf{flow}(c)\})$

Characterisation of Analyses

Direction of information flow

- Forward:
  - $F = \mathsf{flow}(c)$
  - AI_l refers to entry of B^l
  - c has isolated entry
- Backward:
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Quantification over paths

- May:
  - $\bigsqcup = \bigcup$
  - property satisfied by some path
  - interested in least solution (later)
- Must:
  - $\bigsqcup = \bigcap$
  - property satisfied by all paths
  - interested in greatest solution (later)

Heading for a Dataflow Analysis Framework

Roadmap

Goal: solve dataflow equation system by fixpoint iteration

1. Characterise solution of equation system as fixpoint of a transformation

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5. Guarantee termination of fixpoint iteration by ascending chain condition

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5. Guarantee termination of fixpoint iteration by ascending chain condition
6. Optimise fixpoint iteration by worklist algorithm

Heading for a Dataflow Analysis Framework

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$$(\mathsf{AI}_l = \tau_l)_{l \in Lab_c} \iff \Phi((\mathsf{AI}_l)_{l \in Lab_c}) = (\mathsf{AI}_l)_{l \in Lab_c}$$

where $\Phi\left((\mathsf{AI}_l)_{l \in Lab_c}\right) := (\tau_l)_{l \in Lab_c}$

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- **Approach:** approximate fixpoint by iteration

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Order-Theoretic Foundations: The Domain

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Partial Orders

The domain of analysis information usually forms a partial order where the ordering relation compares the “precision” of information.

Definition 3.1 (Partial order)

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

reflexivity: $d_1 \sqsubseteq d_1$
transitivity: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$
antisymmetry: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$

It is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

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Upper Bounds

In the dataflow equation system, analysis information from several predecessors is combined by taking the least upper bound.

Definition 3.3 ((Least) upper bound)

Let $(D, \sqsubseteq)$ be a partial order and $S \subseteq D$.

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$$\bigsqcup\{V_1, \ldots, V_n\} = \bigcup\{V_1, \ldots, V_n\}$$

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3. (Available Expressions) $(D, \sqsubseteq) = (2^{CExp_c}, \supseteq)$. Given $A_1, \ldots, A_n \subseteq CExp_c$,
$$\bigsqcup\{A_1, \ldots, A_n\} = \bigcap\{A_1, \ldots, A_n\}$$

Order-Theoretic Foundations: The Domain

Complete Lattices

Since $\{\varphi_{l'}(\mathrm{AI}_{l'}) \mid (l', l) \in F\}$ can contain arbitrary elements, the existence of least upper bounds must be ensured for arbitrary subsets.

Definition 3.5 (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have least upper bounds. In this case,

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Duality in Complete Lattices

- Dual concept of least upper bound: greatest lower bound
- **Definitions:**
  - An element $d \in D$ is called a lower bound of $S \subseteq D$ if $d \sqsubseteq s$ for every $s \in S$ (notation: $d \sqsubseteq S$).
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- **Lemma:** the following are equivalent:
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- **Lemma:** the following are equivalent:
  - $(D, \sqsubseteq)$ is a complete lattice (i.e., every subset of D has a least upper bound)
  - Every subset of D has a greatest lower bound
- **Corollary:** every complete lattice has a greatest element $\top := \bigsqcap \emptyset$

Chains

Chains are generated by the approximation of the analysis information in the fixpoint iteration.

Definition 3.7 (Chain)

Let $(D, \sqsubseteq)$ be a partial order.

- A subset $S \subseteq D$ is called a chain in D if, for every $d_1, d_2 \in S$,
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- $(D, \sqsubseteq)$ has finite height if all chains are finite. In this case, its height is
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Example 3.8

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1. Every $S \subseteq \mathbb{N}$ is a chain in $(\mathbb{N}, \leq)$ (which is of infinite height)
2. $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots\}$ is a chain in $(2^{\mathbb{N}}, \subseteq)$

Chains

Chains are generated by the approximation of the analysis information in the fixpoint iteration.

Definition 3.7 (Chain)

Let $(D, \sqsubseteq)$ be a partial order.

- A subset $S \subseteq D$ is called a chain in D if, for every $d_1, d_2 \in S$,
$$d_1 \sqsubseteq d_2 \text{ or } d_2 \sqsubseteq d_1$$
(that is, S is a totally ordered subset of D).
- $(D, \sqsubseteq)$ has finite height if all chains are finite. In this case, its height is
$$\max\{|S| \mid S \text{ chain in } D\} - 1.$$

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2. $\{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \ldots\}$ is a chain in $(2^{\mathbb{N}}, \subseteq)$
3. $\{\{0\}, \{1\}, \{2\}\}$ is not a chain in $(2^{\mathbb{N}}, \subseteq)$

The Ascending Chain Condition I

Termination of fixpoint iteration is guaranteed by the following condition.

Definition 3.9 (Ascending Chain Condition)

- A sequence $(d_i)_{i \in \mathbb{N}}$ is called an ascending chain in D if $d_i \sqsubseteq d_{i+1}$ for each $i \in \mathbb{N}$.

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- A partial order $(D, \sqsubseteq)$ satisfies the Ascending Chain Condition (ACC) if each ascending chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ eventually stabilises, i.e., there exists $n \in \mathbb{N}$ such that $d_n = d_{n+1} = \ldots$

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Notes:

- The finite height property implies ACC, but not vice versa (as there might be non-stabilising descending chains – see next slide)
- The complete lattice and ACC properties are orthogonal (see next slide)

The Ascending Chain Condition II

Example 3.10

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Domain requirements for dataflow analysis

$(D, \sqsubseteq)$ must be a complete lattice satisfying ACC