Static Program Analysis

Lecture 16: Abstract Interpretation VI
(Counterexample-Guided Abstraction Refinement)

Winter Semester 2016/17

Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/teaching/ws-1617/spa/

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

13. Nacht der Professoren präsentiert von:

EST. 2006

StudierenOhneGrenzen

am 13.01.2017

ab 22:30 Uhr - im Apollo

NACHT DER PROFESSOREN

Line-Up

◆ DJ Prof. Katoen ◆
Informatik

◆ DJ Prof. Fischer ◆
Biomaterialforschung

◆ DJ Prof. Schröder ◆
Strukturmechanik & Leichtbau

◆ DJ Prof. Böhm ◆
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◆ DJ Prof. Richter ◆
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◆ DJ AIXTRA-Beats ◆
Medizin

& DJ Jiro

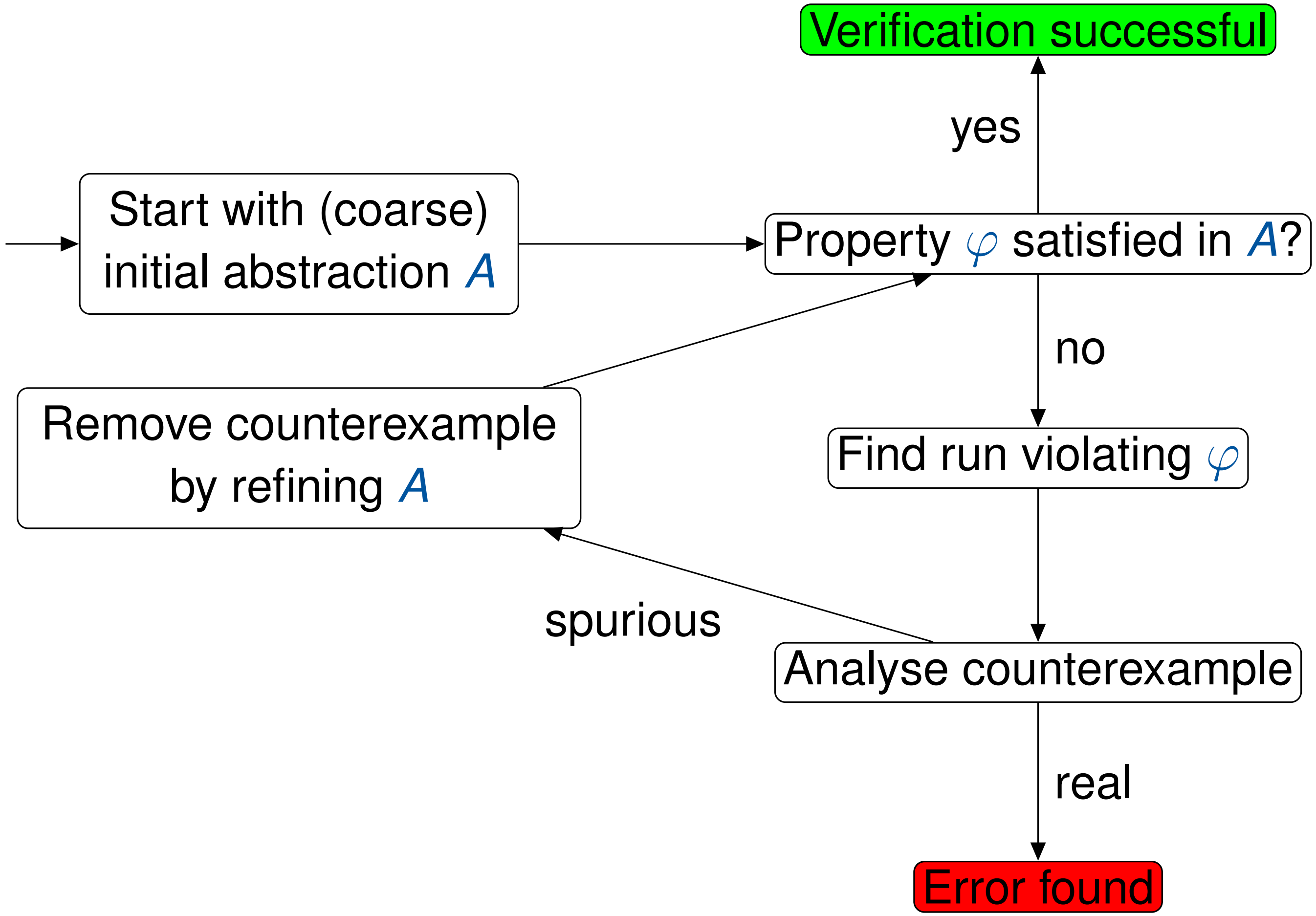
Outline of Lecture 16

Recap: Predicate Abstraction

Computation of Postconditions

Counterexample-Guided Abstraction Refinement

Counterexample-Guided Abstraction Refinement (CEGAR)



Predicate Abstraction I

Definition (Predicate abstraction)

Let Var be a set of variables.

- A predicate is a Boolean expression $p \in BExp$ over Var.
- A state $\sigma \in \Sigma$ satisfies $p \in BExp$ ($\sigma \models p$) if $val_\sigma(p) = \text{true}$.
- p implies q ($p \models q$) if $\sigma \models q$ whenever $\sigma \models p$ (or: p is stronger than q, q is weaker than p).
- p and q are equivalent ($p \equiv q$) if $p \models q$ and $q \models p$.
- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(P) := \left(\left\{\bigwedge Q \mid Q \subseteq P \cup \neg P\right\}, \models\right).$$

Abbreviations: $\text{true} := \bigwedge \emptyset$, $\text{false} := \bigwedge\{p_i, \neg p_i, \ldots\}$

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Predicate Abstraction II

Lemma

$Abs(P)$ *is a complete lattice with*

- $\bot = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2}$ *where* $\overline{b} := \bigwedge\{q \in P \cup \neg P \mid b \models q\}$
  (i.e., strongest formula in $Abs(P)$ *that implies* $Q_1 \wedge Q_2$*)*
- $Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2}$ *(i.e., strongest formula in* $Abs(P)$ *that is implied by* $Q_1 \vee Q_2$*)*

Example

Let $P := \{p_1, p_2, p_3\}$ with $p_1 := (\mathtt{x} = 1)$, $p_2 := (\mathtt{y} = 2)$, $p_1 := (\mathtt{z} = 3)$.

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain

$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge \neg p_2 \wedge p_3$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} = \neg p_2$$

2. For $Q_1 := p_1 \wedge p_2$ and $Q_2 := p_1 \wedge \neg p_2$, we obtain

$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = \text{false}$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{p_1 \wedge (p_2 \vee \neg p_2)} = p_1$$

Predicate Abstraction III

Definition (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\begin{array}{rlcl} & \alpha : 2^\Sigma \to Abs(P) & \text{and} & \gamma : Abs(P) \to 2^\Sigma \\ \text{with} & \alpha(S) := \bigsqcup\{Q_\sigma \mid \sigma \in S\} & \text{and} & \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\} \\ \text{where} & \multicolumn{3}{l}{Q_\sigma := \bigwedge(\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\})} \end{array}$$

Example

- Let $Var := \{\mathtt{x}, \mathtt{y}\}$ and $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x\ <=\ y})$, $p_2 := (\mathtt{x\ =\ y})$, $p_3 := (\mathtt{x\ >\ y})$
- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$, then
$$\begin{aligned} \alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= \underline{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)} \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$
- If $Q = p_1 \wedge \neg p_2 \in Abs(P)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

Abstract Semantics for Predicate Abstraction

Definition (Execution relation for predicate abstraction)

If $c \in Cmd$ and $Q \in Abs(P)$, then $\langle c, Q\rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \mathtt{skip}, Q\rangle \Rightarrow \langle \downarrow, Q\rangle} \qquad \text{(asgn)}\ \frac{}{\langle x := a, Q\rangle \Rightarrow \langle \downarrow, \bigsqcup\{Q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models Q\}\rangle}$$

$$\text{(seq1)}\ \frac{\langle c_1, Q\rangle \Rightarrow \langle c_1', Q'\rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, Q\rangle \Rightarrow \langle c_1' ; c_2, Q'\rangle} \qquad \text{(seq2)}\ \frac{\langle c_1, Q\rangle \Rightarrow \langle \downarrow, Q'\rangle}{\langle c_1 ; c_2, Q\rangle \Rightarrow \langle c_2, Q'\rangle}$$

$$\text{(if1)}\ \frac{}{\langle \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \mathtt{end}, Q\rangle \Rightarrow \langle c_1, \overline{Q \wedge b}\rangle}$$

$$\text{(if2)}\ \frac{}{\langle \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \mathtt{end}, Q\rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b}\rangle}$$

$$\text{(wh1)}\ \frac{}{\langle \mathtt{while}\ b\ \mathtt{do}\ c\ \mathtt{end}, Q\rangle \Rightarrow \langle c ; \mathtt{while}\ b\ \mathtt{do}\ c\ \mathtt{end}, \overline{Q \wedge b}\rangle}$$

$$\text{(wh2)}\ \frac{}{\langle \mathtt{while}\ b\ \mathtt{do}\ c\ \mathtt{end}, Q\rangle \Rightarrow \langle \downarrow, \overline{Q \wedge \neg b}\rangle}$$

Outline of Lecture 16

Computation of Postconditions

Problem: $\overline{b} = \bigwedge\{q \in P \cup \neg P \mid b \models q\}$ (i.e., the strongest formula in $Abs(P)$ that is implied by b) is generally not computable (due to undecidability of implication in certain logics)

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Solutions:

- Over-approximation: fall back to non-strongest postconditions
  - in practice, (automatic) theorem proving
  - for every $i \in \{1, \ldots, n\}$, try to prove $b \models p_i$ and $b \models \neg p_i$
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  - example: Presburger arithmetic (first-order theory of $\mathbb{N}$ with $+$)
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  - example: Presburger arithmetic (first-order theory of $\mathbb{N}$ with $+$)
  - thus $\overline{b}$ computable for WHILE programs without multiplication
- Restriction to finite domains:
  - for example, binary numbers of fixed size
  - thus everything (domain, Galois connection, ...) exactly computable
  - problem: exponential blowup in $n \implies$ solution: Binary Decision Diagrams (BDDs)

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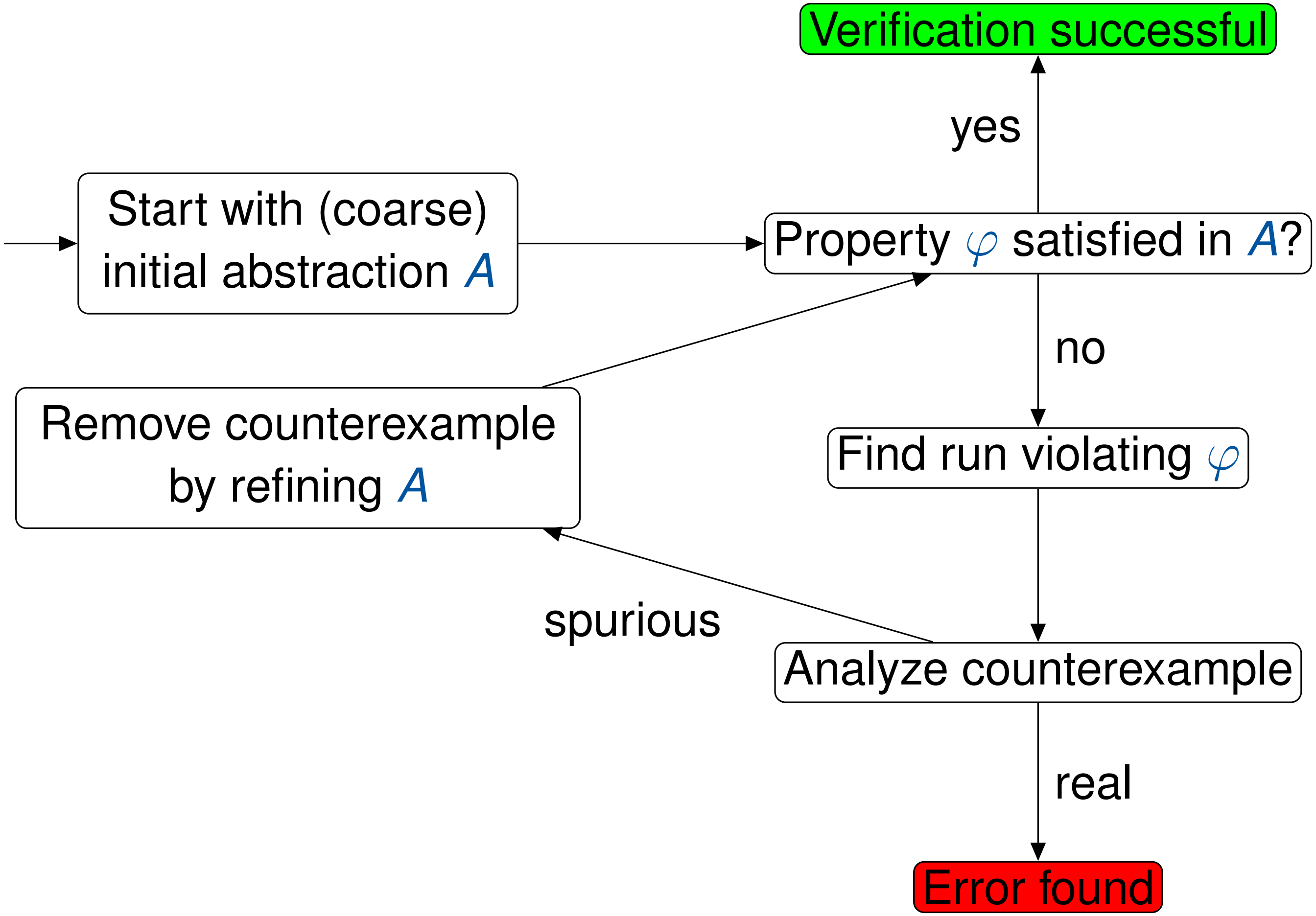

Counterexample-Guided Abstraction Refinement

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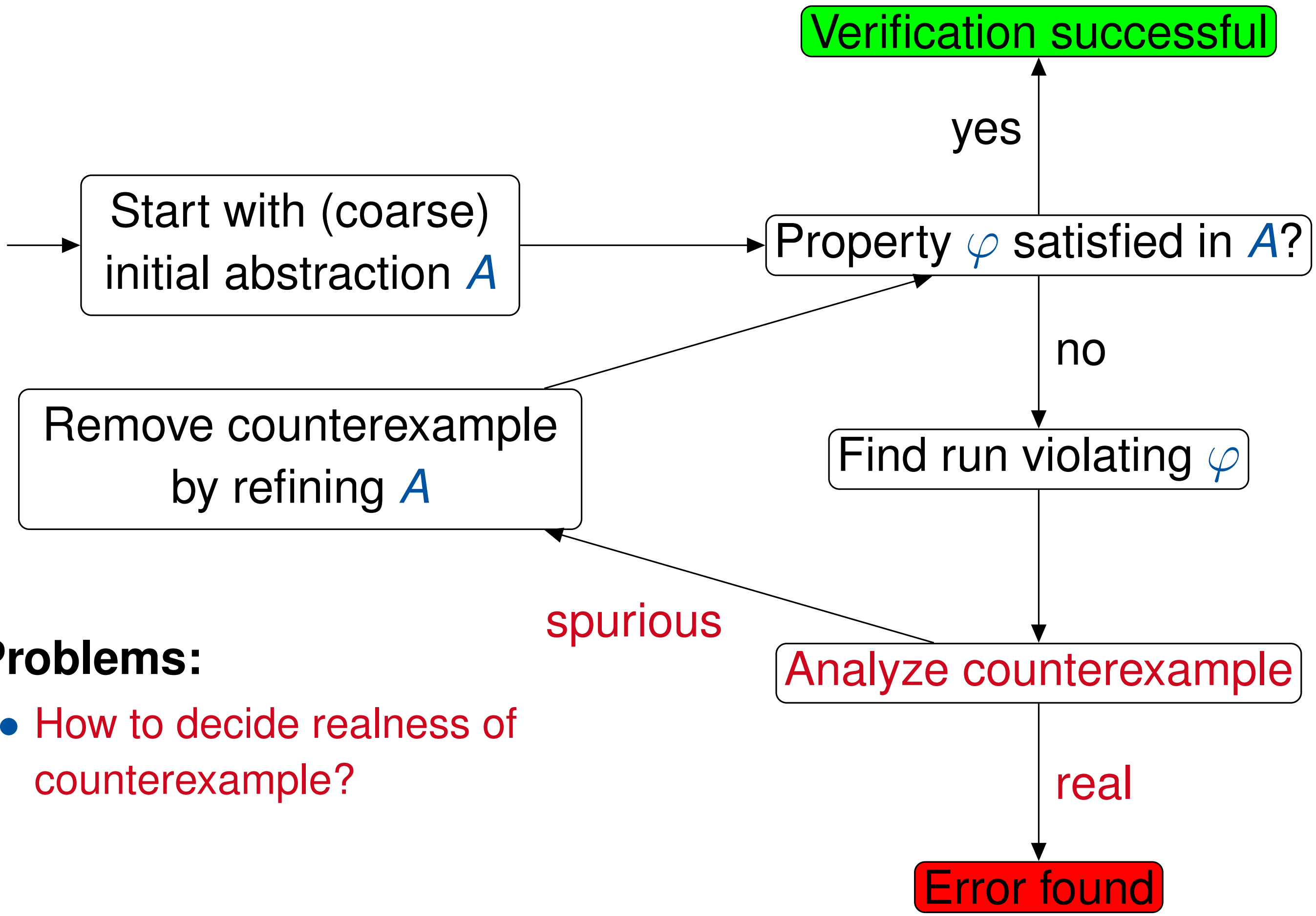

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- How to decide realness of counterexample?

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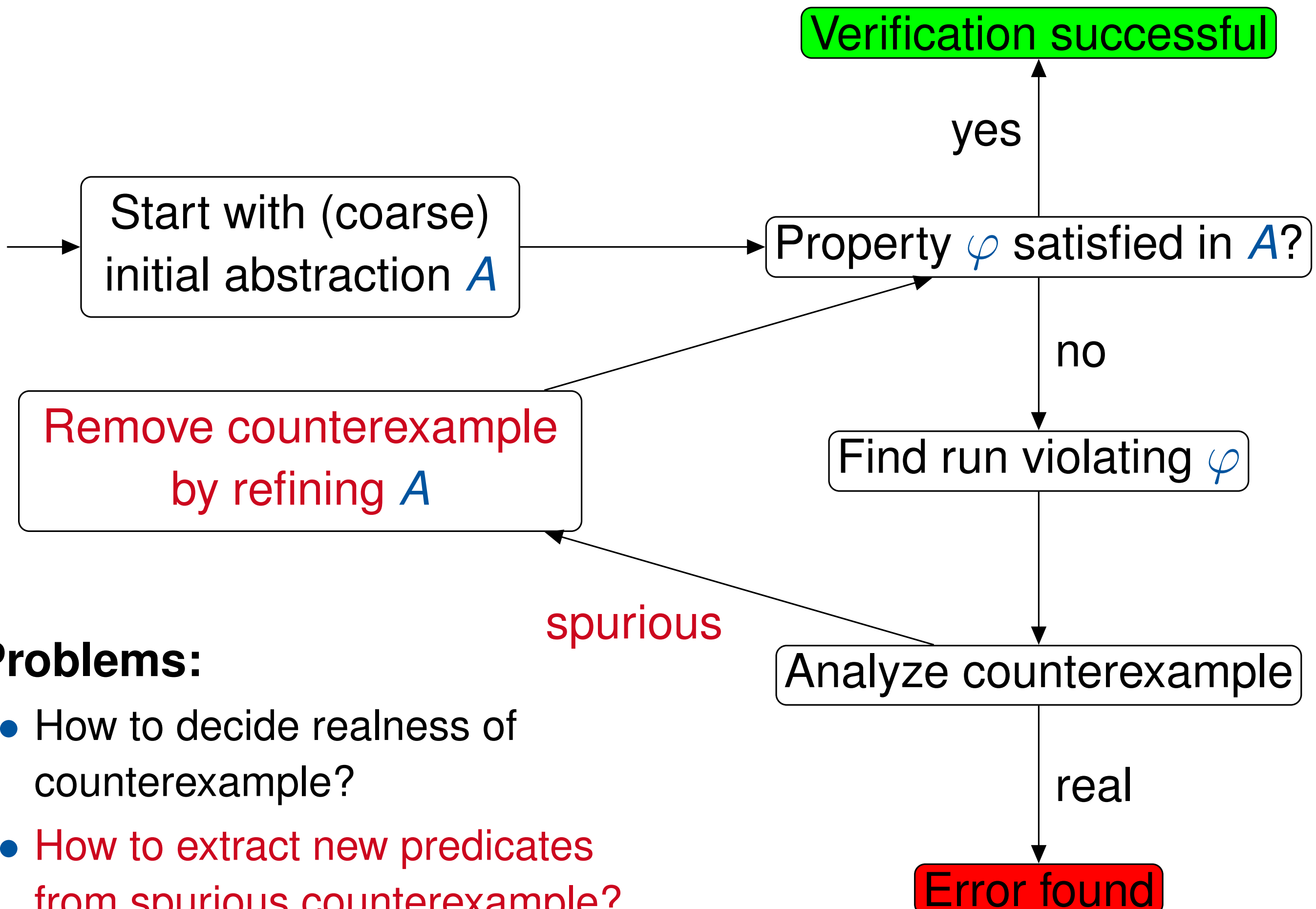

Problems:

- How to decide realness of counterexample?
- How to extract new predicates from spurious counterexample?

Properties of Interest

- A certain program location is not reachable (dead code)
- Division by zero is excluded
- The value of `x` never becomes negative
- After program termination, the value of `y` is even

Counterexample-Guided Abstraction Refinement

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$\Rightarrow$ Counterexample = path to bad locations

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Definition 16.1 (Counterexample)

- A counterexample is a sequence of $k \geq 1$ abstract transitions of the form

$$\langle c_0, \text{true}\rangle \Rightarrow \langle c_1, Q_1\rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k\rangle$$

where

  - $c_0, \ldots, c_k \in Cmd$ (or $c_k = \downarrow$)
  - $Q_1, \ldots, Q_k \in Abs(P)$ with $Q_k \not\equiv \text{false}$

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where
  - $c_0, \ldots, c_k \in Cmd$ (or $c_k = \downarrow$)
  - $Q_1, \ldots, Q_k \in Abs(P)$ with $Q_k \not\models \text{false}$
- It is called real if there exist concrete states $\sigma_0, \ldots, \sigma_k \in \Sigma$ such that
$$\forall i \in \{1, \ldots, k\} : \sigma_i \models Q_i \quad \text{and} \quad \langle c_{i-1}, \sigma_{i-1}\rangle \to \langle c_i, \sigma_i\rangle$$
- Otherwise it is called spurious.

Elimination of Spurious Counterexamples I

Lemma 16.2

If $\langle c_0, \mathrm{true}\rangle \Rightarrow \langle c_1, Q_1\rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k\rangle$ *is a spurious counterexample, there exist Boolean expressions* $b_0, \ldots, b_k$ *with* $b_0 \equiv \mathrm{true}$, $b_k \equiv \mathrm{false}$, *and*

$$\forall i \in \{1, \ldots, k\}, \sigma, \sigma' \in \Sigma : \sigma \models b_{i-1} \wedge \langle c_{i-1}, \sigma\rangle \to \langle c_i, \sigma'\rangle \implies \sigma' \models b_i$$

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Proof (idea).

Inductive definition of b_i as strongest postconditions:

1. $b_0 := \mathrm{true}$
2. for $i = 1, \ldots, k$: definition of b_i depending on b_{i-1} and on (axiom) transition rule applied in $\langle c_{i-1}, .\rangle \Rightarrow \langle c_i, .\rangle$:
   - (skip) $b_i := b_{i-1}$
   - (asgn) $b_i := \exists x'.(b_{i-1}[x \mapsto x'] \wedge x = a[x \mapsto x'])$ (for $x := a$; x' = previous value of x)
   - (if1) $b_i := b_{i-1} \wedge b$
   - (if2) $b_i := b_{i-1} \wedge \neg b$
   - (wh1) $b_i := b_{i-1} \wedge b$
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(yields $b_k \equiv \mathrm{false}$; by induction on k) □

Elimination of Spurious Counterexamples II

Example 16.3

- Let $c_0 := [\texttt{x := z}]^0; [\texttt{z := z + 1}]^1; [\texttt{y := z}]^2;$
  $\texttt{if } [\texttt{x = y}]^3 \texttt{ then } [\texttt{skip}]^4 \texttt{ else } [\texttt{skip}]^5 \texttt{ end}$

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- Forward construction of strongest postconditions:
  - $b_0 := \mathsf{true}$

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    $\implies b_3 := \exists y'.(b_2[\mathtt{y} \mapsto y'] \wedge \mathtt{y} = \mathtt{z}[\mathtt{y} \mapsto y']) \equiv (\mathtt{x} + 1 = \mathtt{z} \wedge \mathtt{y} = \mathtt{z})$
  - (if1) $b_i := b_{i-1} \wedge b$
    $\implies b_4 := (b_3 \wedge \mathtt{x} = \mathtt{y}) \equiv (\mathtt{x} + 1 = \mathtt{z} \wedge \mathtt{y} = \mathtt{z} \wedge \mathtt{x} = \mathtt{y}) \equiv \text{false}$

Abstraction Refinement

- Using $b_1, \ldots, b_{k-1}$ as computed before, let $P' := P \cup \{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n$ are the atomic conjuncts occurring in $b_1, \ldots, b_{k-1}$
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Abstraction Refinement

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- Refine $Abs(P)$ to $Abs(P')$

Lemma 16.4

After refinement, the spurious counterexample

$$\langle c_0, \mathrm{true}\rangle \Rightarrow \langle c_1, Q_1\rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k\rangle$$

with $Q_k \not\equiv \mathrm{false}$ *does not exist anymore.*

Proof.

omitted $\square$

A Simple Example

Example 16.5 (cf. Example 16.3)

- Let $c_0 := [\mathtt{x\ :=\ z}]^0; [\mathtt{z\ :=\ z\ +\ 1}]^1; [\mathtt{y\ :=\ z}]^2;$
  $\mathtt{if}\ [\mathtt{x\ =\ y}]^3\ \mathtt{then}\ [\mathtt{skip}]^4\ \mathtt{else}\ [\mathtt{skip}]^5\ \mathtt{end}$
- $P = \emptyset$, $P' = \{\underbrace{\mathtt{x} = \mathtt{z}}_{p_1}, \underbrace{\mathtt{x} + 1 = \mathtt{z}}_{p_2}, \underbrace{\mathtt{y} = \mathtt{z}}_{p_3}\}$

A Simple Example

Example 16.5 (cf. Example 16.3)

- Let $c_0 := [\texttt{x := z}]^0;[\texttt{z := z + 1}]^1;[\texttt{y := z}]^2;$
  $\texttt{if } [\texttt{x = y}]^3 \texttt{ then } [\texttt{skip}]^4 \texttt{ else } [\texttt{skip}]^5 \texttt{ end}$
- $P = \emptyset$, $P' = \{\underbrace{\texttt{x} = \texttt{z}}_{p_1}, \underbrace{\texttt{x} + 1 = \texttt{z}}_{p_2}, \underbrace{\texttt{y} = \texttt{z}}_{p_3}\}$
- Refined abstract transitions:

$$\langle 0, \mathsf{true}\rangle$$

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- Refined abstract transitions:

$$\langle 0, \mathsf{true}\rangle \Rightarrow \langle 1, p_1 \wedge \neg p_2\rangle$$

A Simple Example

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- Let $c_0 := [\texttt{x := z}]^0;[\texttt{z := z + 1}]^1;[\texttt{y := z}]^2;$
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- Refined abstract transitions:

$$\begin{aligned}\langle 0, \mathsf{true}\rangle &\Rightarrow \langle 1, p_1 \wedge \neg p_2\rangle \\ &\Rightarrow \langle 2, \neg p_1 \wedge p_2\rangle\end{aligned}$$

A Simple Example

Example 16.5 (cf. Example 16.3)

- Let $c_0 := [\texttt{x := z}]^0;[\texttt{z := z + 1}]^1;[\texttt{y := z}]^2;$
  $\texttt{if } [\texttt{x = y}]^3 \texttt{ then } [\texttt{skip}]^4 \texttt{ else } [\texttt{skip}]^5 \texttt{ end}$
- $P = \emptyset$, $P' = \{\underbrace{\texttt{x} = \texttt{z}}_{p_1}, \underbrace{\texttt{x} + 1 = \texttt{z}}_{p_2}, \underbrace{\texttt{y} = \texttt{z}}_{p_3}\}$
- Refined abstract transitions:

$$\begin{aligned}\langle 0, \mathrm{true}\rangle &\Rightarrow \langle 1, p_1 \wedge \neg p_2\rangle\\ &\Rightarrow \langle 2, \neg p_1 \wedge p_2\rangle\\ &\Rightarrow \langle 3, \neg p_1 \wedge p_2 \wedge p_3\rangle\end{aligned}$$

A Simple Example

Example 16.5 (cf. Example 16.3)

- Let $c_0 := [\mathtt{x\ :=\ z}]^0; [\mathtt{z\ :=\ z\ +\ 1}]^1; [\mathtt{y\ :=\ z}]^2;$
  $\mathtt{if}\ [\mathtt{x\ =\ y}]^3\ \mathtt{then}\ [\mathtt{skip}]^4\ \mathtt{else}\ [\mathtt{skip}]^5\ \mathtt{end}$
- $P = \emptyset$, $P' = \{\underbrace{\mathtt{x} = \mathtt{z}}_{p_1}, \underbrace{\mathtt{x} + 1 = \mathtt{z}}_{p_2}, \underbrace{\mathtt{y} = \mathtt{z}}_{p_3}\}$
- Refined abstract transitions:

$$\begin{aligned}\langle 0, \mathsf{true}\rangle &\Rightarrow \langle 1, p_1 \wedge \neg p_2\rangle \\ &\Rightarrow \langle 2, \neg p_1 \wedge p_2\rangle \\ &\Rightarrow \langle 3, \neg p_1 \wedge p_2 \wedge p_3\rangle \\ &\Rightarrow \langle 4, \underbrace{\overline{\neg p_1 \wedge p_2 \wedge p_3 \wedge \mathtt{x{=}y}}}_{\equiv\ \mathsf{false}}\rangle\end{aligned}$$

Counterexample-Guided Abstraction Refinement

Another Example: Multiplication

Example 16.6

- Let $c_0 :=$ $[\mathtt{z\ :=\ 0}]^0$;

```
while [x > 0]^1 do
  [z := z + y]^2;
  [x := x - 1]^3
end;
if [z mod y = 0]^4 then
  [skip]^5
else
  [skip]^6
end;
```

- Global assumption: $\mathtt{y} > 0$
- Interesting property: label 6 unreachable (since $\mathtt{z}$ multiple of $\mathtt{y}$)

Another Example: Multiplication

Example 16.6

- Let $c_0 :=$

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[z := 0]^0;
while [x > 0]^1 do
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  [skip]^5
else
  [skip]^6
end;
```

- Global assumption: $\mathtt{y} > 0$
- Interesting property: label 6 unreachable (since z multiple of y)
- Initial abstraction: $P = \emptyset$ ($\implies Abs(P) = \{\mathsf{true}, \mathsf{false}\}$)
- Abstraction refinement: on the board