Static Program Analysis

Lecture 16: Abstract Interpretation VI (Counterexample-Guided Abstraction Refinement)

Winter Semester 2016/17

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https://moves.rwth-aachen.de/teaching/ws-1617/spa/

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

13. Nacht der Professoren präsentiert von:

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am 13.01.2017
ab 22:30 Uhr - im Apollo

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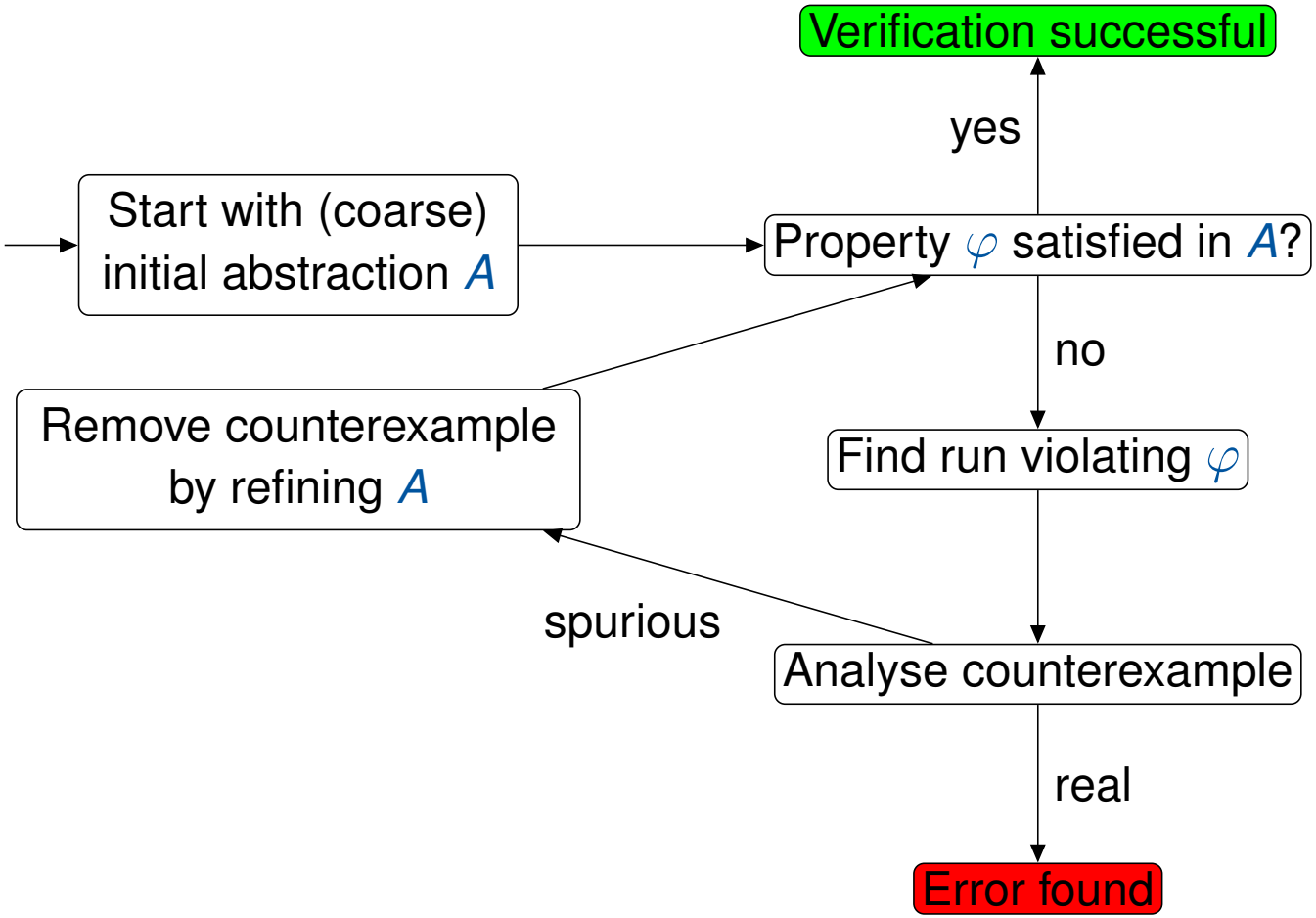
Recap: Predicate Abstraction

Counterexample-Guided Abstraction Refinement (CEGAR)



Predicate Abstraction I

Definition (Predicate abstraction)

Let Var be a set of variables.

- A predicate is a Boolean expression $p \in BExp$ over Var.
- A state $\sigma \in \Sigma$ satisfies $p \in BExp$ ($\sigma \models p$) if $val_\sigma(p) = \text{true}$.
- p implies q ($p \models q$) if $\sigma \models q$ whenever $\sigma \models p$ (or: p is stronger than q, q is weaker than p).
- p and q are equivalent ($p \equiv q$) if $p \models q$ and $q \models p$.
- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(P) := \left(\left\{ \bigwedge Q \mid Q \subseteq P \cup \neg P \right\}, \models \right).$$

Abbreviations: $\text{true} := \bigwedge \emptyset$, $\text{false} := \bigwedge \{p_i, \neg p_i, \ldots\}$

Predicate Abstraction II

Lemma

$Abs(P)$ is a complete lattice with

- $\bot = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2}$ *where* $\overline{b} := \bigwedge\{q \in P \cup \neg P \mid b \models q\}$
  (i.e., strongest formula in $Abs(P)$ that implies $Q_1 \wedge Q_2$)
- $Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2}$ *(i.e., strongest formula in $Abs(P)$ that is implied by $Q_1 \vee Q_2$)*

Example

Let $P := \{p_1, p_2, p_3\}$ with $p_1 := (\mathrm{x} = 1)$, $p_2 := (\mathrm{y} = 2)$, $p_1 := (\mathrm{z} = 3)$.

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain
$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge \neg p_2 \wedge p_3$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} = \neg p_2$$
2. For $Q_1 := p_1 \wedge p_2$ and $Q_2 := p_1 \wedge \neg p_2$, we obtain
$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = \text{false}$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{p_1 \wedge (p_2 \vee \neg p_2)} = p_1$$

Predicate Abstraction III

Definition (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\begin{array}{rlll} & \alpha : 2^{\Sigma} \to \mathit{Abs}(P) & \text{and} & \gamma : \mathit{Abs}(P) \to 2^{\Sigma} \\ \text{with} & \alpha(S) := \bigsqcup\{Q_\sigma \mid \sigma \in S\} & \text{and} & \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\} \\ \text{where} & Q_\sigma := \bigwedge(\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\}) & & \end{array}$$

Example

- Let $\mathit{Var} := \{\mathtt{x}, \mathtt{y}\}$ and $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x} <= \mathtt{y})$, $p_2 := (\mathtt{x} = \mathtt{y})$, $p_3 := (\mathtt{x} > \mathtt{y})$
- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$, then
$$\begin{aligned} \alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= \underline{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)} \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$
- If $Q = p_1 \wedge \neg p_2 \in \mathit{Abs}(P)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

Abstract Semantics for Predicate Abstraction

Definition (Execution relation for predicate abstraction)

If $c \in Cmd$ and $Q \in Abs(P)$, then $\langle c, Q \rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, Q \rangle \Rightarrow \langle \downarrow, Q \rangle} \qquad \text{(asgn)}\ \frac{}{\langle x := a, Q \rangle \Rightarrow \langle \downarrow, \bigsqcup \{ Q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models Q \} \rangle}$$

$$\text{(seq1)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle c_1', Q' \rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_1' ; c_2, Q' \rangle} \qquad \text{(seq2)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle \downarrow, Q' \rangle}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_2, Q' \rangle}$$

$$\text{(if1)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, Q \rangle \Rightarrow \langle c_1, \overline{Q \wedge b} \rangle}$$

$$\text{(if2)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, Q \rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b} \rangle}$$

$$\text{(wh1)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, Q \rangle \Rightarrow \langle c ; \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \overline{Q \wedge b} \rangle}$$

$$\text{(wh2)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, Q \rangle \Rightarrow \langle \downarrow, \overline{Q \wedge \neg b} \rangle}$$

Computation of Postconditions

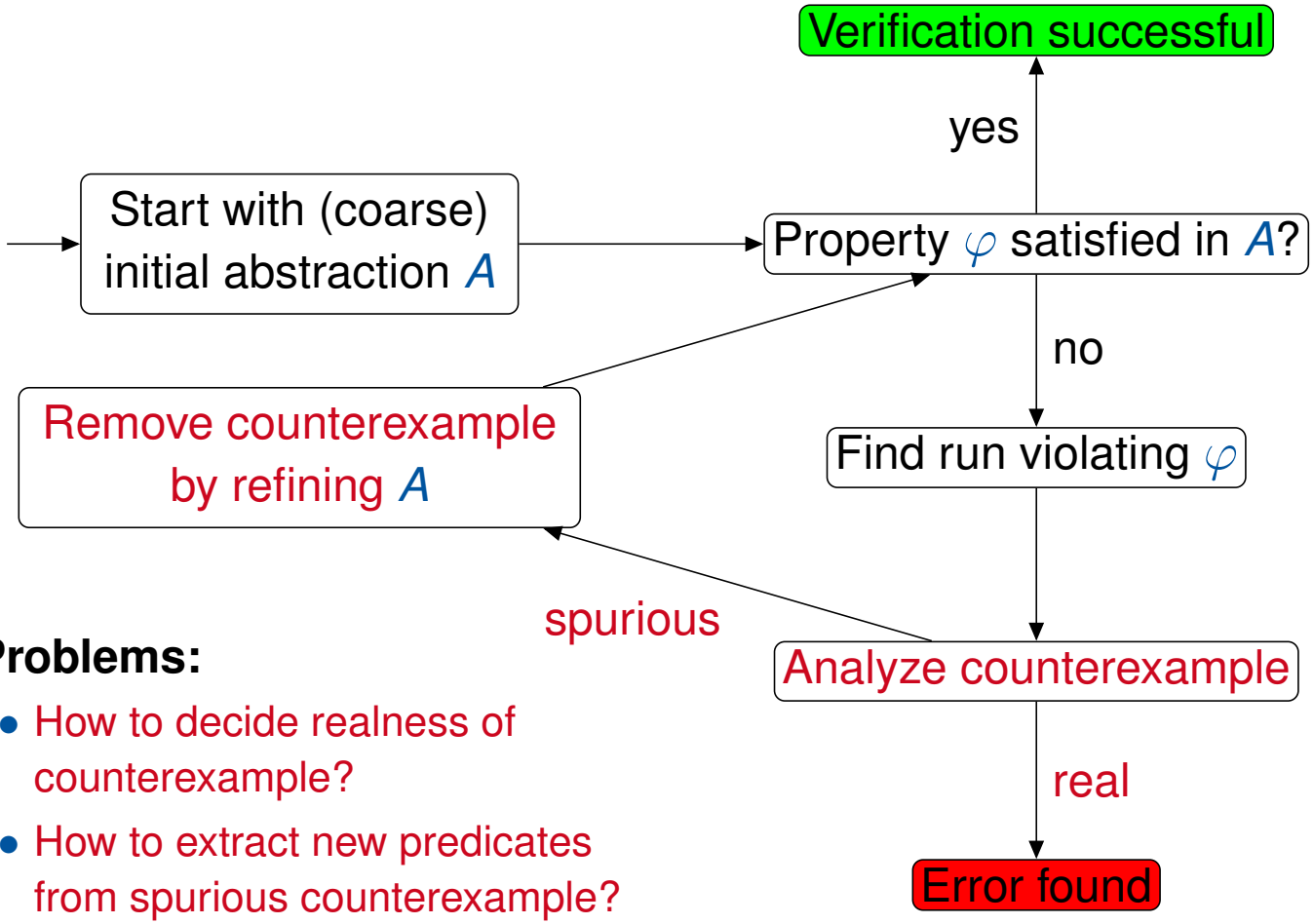
Problem: $\overline{b} = \bigwedge \{q \in P \cup \neg P \mid b \models q\}$ (i.e., the strongest formula in $Abs(P)$ that is implied by b) is generally not computable (due to undecidability of implication in certain logics)

Solutions:

- Over-approximation: fall back to non-strongest postconditions
  - in practice, (automatic) theorem proving
  - for every $i \in \{1, \ldots, n\}$, try to prove $b \models p_i$ and $b \models \neg p_i$
  - approximate $\overline{b}$ by conjunction of all provable literals
- Restriction of programs:
  - $\models$ decidable for certain logics
  - example: Presburger arithmetic (first-order theory of $\mathbb{N}$ with $+$)
  - thus $\overline{b}$ computable for WHILE programs without multiplication
- Restriction to finite domains:
  - for example, binary numbers of fixed size
  - thus everything (domain, Galois connection, ...) exactly computable
  - problem: exponential blowup in n $\Longrightarrow$ solution: Binary Decision Diagrams (BDDs)

Counterexample-Guided Abstraction Refinement

Reminder: CEGAR



Problems:

- How to decide realness of counterexample?
- How to extract new predicates from spurious counterexample?

Properties of Interest

- A certain program location is not reachable (dead code)
- Division by zero is excluded
- The value of `x` never becomes negative
- After program termination, the value of `y` is even

$\Rightarrow$ All representable as (non-)reachability of "bad locations"

$\Rightarrow$ Counterexample = path to bad locations

Definition 16.1 (Counterexample)

- A counterexample is a sequence of $k \geq 1$ abstract transitions of the form

$$\langle c_0, \text{true} \rangle \Rightarrow \langle c_1, Q_1 \rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k \rangle$$

  where
  - $c_0, \ldots, c_k \in \mathit{Cmd}$ (or $c_k = \downarrow$)
  - $Q_1, \ldots, Q_k \in \mathit{Abs}(P)$ with $Q_k \not\equiv \text{false}$
- It is called real if there exist concrete states $\sigma_0, \ldots, \sigma_k \in \Sigma$ such that

$$\forall i \in \{1, \ldots, k\} : \sigma_i \models Q_i \quad \text{and} \quad \langle c_{i-1}, \sigma_{i-1} \rangle \rightarrow \langle c_i, \sigma_i \rangle$$

- Otherwise it is called spurious.

Elimination of Spurious Counterexamples I

Lemma 16.2

If $\langle c_0, \text{true}\rangle \Rightarrow \langle c_1, Q_1\rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k\rangle$ *is a spurious counterexample, there exist Boolean expressions* $b_0, \ldots, b_k$ *with* $b_0 \equiv \text{true}$, $b_k \equiv \text{false}$, *and*

$$\forall i \in \{1, \ldots, k\}, \sigma, \sigma' \in \Sigma : \sigma \models b_{i-1} \wedge \langle c_{i-1}, \sigma\rangle \to \langle c_i, \sigma'\rangle \implies \sigma' \models b_i$$

Proof (idea).

Inductive definition of b_i as strongest postconditions:

1. $b_0 := \text{true}$
2. for $i = 1, \ldots, k$: definition of b_i depending on b_{i-1} and on (axiom) transition rule applied in $\langle c_{i-1}, .\rangle \Rightarrow \langle c_i, .\rangle$:
   - (skip) $b_i := b_{i-1}$
   - (asgn) $b_i := \exists x'.(b_{i-1}[x \mapsto x'] \wedge x = a[x \mapsto x'])$
     (for $x := a$; x' = previous value of x)
   - (if1) $b_i := b_{i-1} \wedge b$
   - (if2) $b_i := b_{i-1} \wedge \neg b$
   - (wh1) $b_i := b_{i-1} \wedge b$
   - (wh2) $b_i := b_{i-1} \wedge \neg b$

(yields $b_k \equiv \text{false}$; by induction on k) □

Elimination of Spurious Counterexamples II

Example 16.3

- Let $c_0 := [\mathtt{x := z}]^0; [\mathtt{z := z + 1}]^1; [\mathtt{y := z}]^2;$
  $\mathtt{if}\ [\mathtt{x = y}]^3\ \mathtt{then}\ [\mathtt{skip}]^4\ \mathtt{else}\ [\mathtt{skip}]^5\ \mathtt{end}$
- Interesting property: after termination, $\mathtt{x} \neq \mathtt{y}$, i.e., label 4 unreachable
- Initial abstraction: $P = \emptyset$ ($\Longrightarrow$ $Abs(P) = \{\text{true}, \text{false}\}$)
- (Spurious) counterexample: $\langle 0, \text{true}\rangle \Rightarrow \langle 1, \text{true}\rangle \Rightarrow \langle 2, \text{true}\rangle \Rightarrow \langle 3, \text{true}\rangle \Rightarrow \langle 4, \text{true}\rangle$
- Forward construction of strongest postconditions:
  - $b_0 := \text{true}$
  - (asgn) $b_i := \exists x'.(b_{i-1}[x \mapsto x'] \wedge x = a[x \mapsto x'])$
    $\Longrightarrow$ $b_1 := \exists x'.(b_0[\mathtt{x} \mapsto x'] \wedge \mathtt{x} = \mathtt{z}[\mathtt{x} \mapsto x']) \equiv (\mathtt{x} = \mathtt{z})$
  - (asgn) $b_i := \exists x'.(b_{i-1}[x \mapsto x'] \wedge x = a[x \mapsto x'])$
    $\Longrightarrow$ $b_2 := \exists z'.(b_1[\mathtt{z} \mapsto z'] \wedge \mathtt{z} = \mathtt{z} + 1[\mathtt{z} \mapsto z'])$
    $= \exists z'.(\mathtt{x} = z' \wedge \mathtt{z} = z' + 1) \equiv (\mathtt{x} + 1 = \mathtt{z})$
  - (asgn) $b_i := \exists x'.(b_{i-1}[x \mapsto x'] \wedge x = a[x \mapsto x'])$
    $\Longrightarrow$ $b_3 := \exists y'.(b_2[\mathtt{y} \mapsto y'] \wedge \mathtt{y} = \mathtt{z}[\mathtt{y} \mapsto y']) \equiv (\mathtt{x} + 1 = \mathtt{z} \wedge \mathtt{y} = \mathtt{z})$
  - (if1) $b_i := b_{i-1} \wedge b$
    $\Longrightarrow$ $b_4 := (b_3 \wedge \mathtt{x} = \mathtt{y}) \equiv (\mathtt{x} + 1 = \mathtt{z} \wedge \mathtt{y} = \mathtt{z} \wedge \mathtt{x} = \mathtt{y}) \equiv \text{false}$

Abstraction Refinement

- Using $b_1, \ldots, b_{k-1}$ as computed before, let $P' := P \cup \{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n$ are the atomic conjuncts occurring in $b_1, \ldots, b_{k-1}$
- Refine $Abs(P)$ to $Abs(P')$

Lemma 16.4

After refinement, the spurious counterexample

$$\langle c_0, \text{true} \rangle \Rightarrow \langle c_1, Q_1 \rangle \Rightarrow \ldots \Rightarrow \langle c_k, Q_k \rangle$$

with $Q_k \not\equiv \text{false}$ *does not exist anymore.*

Proof.

omitted □

A Simple Example

Example 16.5 (cf. Example 16.3)

- Let $c_0 := [\texttt{x := z}]^0; [\texttt{z := z + 1}]^1; [\texttt{y := z}]^2;$
  $\texttt{if } [\texttt{x = y}]^3 \texttt{ then } [\texttt{skip}]^4 \texttt{ else } [\texttt{skip}]^5 \texttt{ end}$
- $P = \emptyset$, $P' = \{\underbrace{\texttt{x} = \texttt{z}}_{p_1}, \underbrace{\texttt{x} + 1 = \texttt{z}}_{p_2}, \underbrace{\texttt{y} = \texttt{z}}_{p_3}\}$
- Refined abstract transitions:

$$\begin{aligned}\langle 0, \text{true}\rangle &\Rightarrow \langle 1, p_1 \wedge \neg p_2\rangle \\ &\Rightarrow \langle 2, \neg p_1 \wedge p_2\rangle \\ &\Rightarrow \langle 3, \neg p_1 \wedge p_2 \wedge p_3\rangle \\ &\Rightarrow \langle 4, \underbrace{\overline{\neg p_1 \wedge p_2 \wedge p_3 \wedge \texttt{x=y}}}_{\equiv \text{ false}}\rangle\end{aligned}$$

Another Example: Multiplication

Example 16.6

- Let $c_0 :=$
```
[z := 0]^0;
while [x > 0]^1 do
  [z := z + y]^2;
  [x := x - 1]^3
end;
if [z mod y = 0]^4 then
  [skip]^5
else
  [skip]^6
end;
```
- Global assumption: $y > 0$
- Interesting property: label 6 unreachable (since z multiple of y)
- Initial abstraction: $P = \emptyset$ ($\Longrightarrow$ $Abs(P) = \{\text{true}, \text{false}\}$)
- Abstraction refinement: on the board