



Static Program Analysis

Lecture 15: Abstract Interpretation V (Numerical & Predicate Abstraction)

Winter Semester 2016/17

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<https://moves.rwth-aachen.de/teaching/ws-1617/spa/>

Overview of Numerical Abstraction Domains

Outline of Lecture 15

Overview of Numerical Abstraction Domains

Overview of Abstraction Refinement Using Predicates

Predicate Abstraction

Abstract Semantics for Predicate Abstraction

Overview of Numerical Abstraction Domains

Non-Relational Abstraction Domains

Here, abstract values are independently referring to single variables:

Example 15.1 (Non-relational domains)

- **Signs** (cf. Example 11.3): $\text{sgn}(x) = s$ ($x \in \text{Var}$, $s \in \{+, -, 0\}$)
- **Intervals** (cf. Example 11.4): $x \in J$ ($x \in \text{Var}$, $J \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \cup \{\emptyset\}$)
- **Parities** (cf. Example 11.2): $x \in \mathbb{Z}_p$ ($x \in \text{Var}$, $p \in \{\text{even}, \text{odd}\}$)
- **Congruences** (cf. Lemma 14.2): $x \bmod m = k$ ($x \in \text{Var}$, $m > 1$, $k \in \{0, \dots, m - 1\}$)

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Observations

- Expressive power:
 - Signs $<$ Intervals (since $+ \cong [1, +\infty]$, ...)
 - Parities $<$ Congruences (since $x \text{ even} \iff x \bmod 2 = 0$, ...)
 - Intervals and Congruences are “incomparable”
- Congruences can prove disequalities (“ $x \neq y$ ”) but not inequalities (“ $x \leq y$ ”)
 - e.g., $x \bmod m = k_x, y \bmod m = k_y, k_x \neq k_y \implies$ no zero division in $1/(x - y)$
 - but $x \leq y$ generally not representable
- Non-relational domains efficient to represent and manipulate

Overview of Numerical Abstraction Domains

Relational Abstraction Domains

Here, interdependencies between variables are captured:

Example 15.2 (Relational domains)

- **Difference Bound Matrices (DBMs)**: conjunctions of $x - y \leq c$ and $\pm x \leq c$ ($x, y \in Var, c \in \mathbb{Z}$)
- **Octagons**: conjunctions of $ax + by \leq c$ ($x, y \in Var, a, b \in \{-1, 0, 1\}, c \in \mathbb{Z}$)
- **Octahedra**: conjunctions of $a_1x_1 + \dots + a_nx_n \leq c$ ($x_i \in Var, a_i \in \{-1, 0, 1\}, c \in \mathbb{Z}$)
- **Polyhedra**: conjunctions of $a_1x_1 + \dots + a_nx_n \leq c$ ($x_i \in Var, a_i \in \mathbb{Z}, c \in \mathbb{Z}$)

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Observations

- Expressive power:
 - DBMs < Octagons < Octahedra < Polyhedra
 - Intervals < DBMs (since $x \in [c_1, c_2] \iff -x \leq -c_1 \wedge x \leq c_2$)
- Can prove inequalities but not (general) disequalities
- Representation and manipulation generally more involved
 - Polyhedra require computation of convex hulls (exponential in $|Var|$)

Overview of Numerical Abstraction Domains

Combining Non-Relational and Relational Domains

Linear Congruences combine features of Congruences and Polyhedra:

- Given by conjunctions of

$$(a_1x_1 + \dots + a_nx_n) \bmod m = z$$

$$(x_i \in \text{Var}, a_i \in \mathbb{Z}, m > 1, z \in \mathbb{Z})$$

- Typical application:

$$2x + 1 \bmod m = k_x, y \bmod m = k_y, k_x \neq k_y \implies \text{no zero division in } 1/(2x + 1 - y)$$

- Again usable for proving disequalities but not inequalities

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 - **predicate abstraction** and
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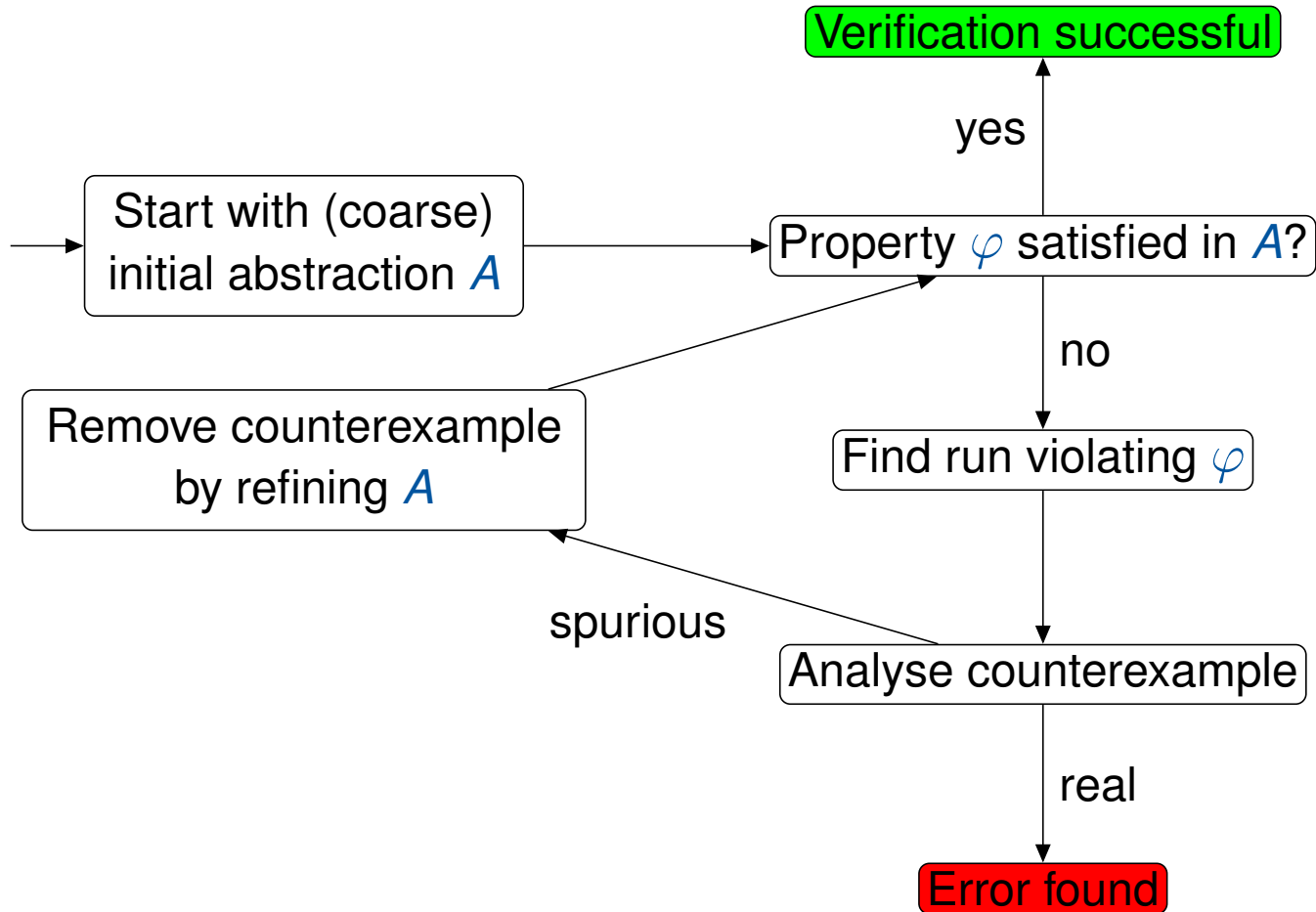
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- **Abstraction refinement:** most successful (automatic) method based on
 - **predicate abstraction** and
 - analysing **counterexamples**
- **Difference** to standard abstract interpretation with fixed domain:
abstraction **parametrised by and specific to program**

Overview of Abstraction Refinement Using Predicates

Counterexample-Guided Abstraction Refinement (CEGAR)



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Abstraction Refinement for Predicates

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2. Use **Galois connection** that classifies program states according to validity of predicates (**predicate abstraction**)
3. Compute new **abstract semantics** and search for new **counterexamples**
4. **Iterate** until property satisfied or real counterexample found (with increasing set of predicates; can entail non-termination)

Predicate Abstraction

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Definition 15.3 (Predicate abstraction)

Let Var be a set of variables.

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- p and q are **equivalent** ($p \equiv q$) if $p \models q$ and $q \models p$.
- Let $P = \{p_1, \dots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \dots, \neg p_n\}$.
An element of $P \cup \neg P$ is called a **literal**. The **predicate abstraction lattice** is defined by:

$$Abs(P) := \left(\left\{ \bigwedge Q \mid Q \subseteq P \cup \neg P \right\}, \models \right).$$

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Abbreviations: $\text{true} := \bigwedge \emptyset$, $\text{false} := \bigwedge \{p_i, \neg p_i, \dots\}$

Predicate Abstraction

Predicate Abstraction II

Lemma 15.4

$Abs(P)$ is a *complete lattice* with

- $\perp = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2}$ where $\bar{b} := \bigwedge \{q \in P \cup \neg P \mid b \models q\}$
(i.e., strongest formula in $Abs(P)$ that is implied by Q_1 and Q_2)
- $Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2}$ (i.e., strongest formula in $Abs(P)$ that is implied by $Q_1 \vee Q_2$)

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Example 15.5

Let $P := \{p_1, p_2, p_3\}$ with $p_1 := (x = 1)$, $p_2 := (y = 2)$, $p_3 := (z = 3)$.

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain

$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = \overline{p_1 \wedge \neg p_2 \wedge p_3}$$

$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} = \neg p_2$$

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Predicate Abstraction III

Important: if predicates are interdependent, then generally

1. $Q_1 \sqcap Q_2 (= \overline{Q_1 \wedge Q_2}) \neq \wedge(Q_1 \cup Q_2)$ (but $\equiv \wedge(Q_1 \cup Q_2)$)

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1. – $p_1 := (x \leq y), p_2 := (x \geq y), p_3 := (x = y)$
– $Q_1 := p_1, Q_2 := p_2$
 $\Rightarrow Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge p_2 \wedge p_3 \not\equiv \wedge(Q_1 \cup Q_2) = p_1 \wedge p_2$
2. – $p_1 := (x > y), p_2 := (x \geq y), p_3 := (x = y)$
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If these dependencies are ignored, then (computationally simpler) **Cartesian Abstraction** is performed.

Predicate Abstraction

Predicate Abstraction IV

Definition 15.7 (Galois connection for predicate abstraction)

The **Galois connection for predicate abstraction** is determined by

$$\begin{array}{l} \alpha : 2^\Sigma \rightarrow Abs(P) \quad \text{and} \quad \gamma : Abs(P) \rightarrow 2^\Sigma \\ \text{with} \quad \alpha(S) := \bigsqcup \{Q_\sigma \mid \sigma \in S\} \quad \text{and} \quad \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\} \\ \text{where} \quad Q_\sigma := \bigwedge (\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\}) \end{array}$$

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$$\begin{aligned} \text{then } \alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= \frac{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)}{} \\ &= \frac{(p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3)}{} \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$

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then $\alpha(S) = Q_{\sigma_1} \sqcup Q_{\sigma_2}$
$$\begin{aligned} &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3) \\ &= \frac{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)}{\equiv} \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$
- If $Q = p_1 \wedge \neg p_2 \in \text{Abs}(P)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(x) < \sigma(y)\}$

Abstract Semantics for Predicate Abstraction

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Definition 15.9 (Execution relation for predicate abstraction)

If $c \in \text{Cmd}$ and $Q \in \text{Abs}(P)$, then $\langle c, Q \rangle$ is called an **abstract configuration**. The **execution relation for predicate abstraction** is defined by the following rules:

$$\begin{array}{c} \text{(skip)} \frac{}{\langle \text{skip}, Q \rangle \Rightarrow \langle \downarrow, Q \rangle} \quad \text{(asgn)} \frac{}{\langle x := a, Q \rangle \Rightarrow \langle \downarrow, \bigsqcup \{ Q_{\sigma[x \mapsto \text{val}_{\sigma}(a)]} \mid \sigma \models Q \} \rangle} \\ \text{(seq1)} \frac{\langle c_1, Q \rangle \Rightarrow \langle c'_1, Q' \rangle \quad c'_1 \neq \downarrow}{\langle c_1; c_2, Q \rangle \Rightarrow \langle c'_1; c_2, Q' \rangle} \quad \text{(seq2)} \frac{\langle c_1, Q \rangle \Rightarrow \langle \downarrow, Q' \rangle}{\langle c_1; c_2, Q \rangle \Rightarrow \langle c_2, Q' \rangle} \\ \text{(if1)} \frac{}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end}, Q \rangle \Rightarrow \langle c_1, \overline{Q \wedge b} \rangle} \\ \text{(if2)} \frac{}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end}, Q \rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b} \rangle} \\ \text{(wh1)} \frac{}{\langle \text{while } b \text{ do } c \text{ end}, Q \rangle \Rightarrow \langle c; \text{while } b \text{ do } c \text{ end}, \overline{Q \wedge b} \rangle} \\ \text{(wh2)} \frac{}{\langle \text{while } b \text{ do } c \text{ end}, Q \rangle \Rightarrow \langle \downarrow, \overline{Q \wedge \neg b} \rangle} \end{array}$$

Abstract Semantics for Predicate Abstraction

Remarks

- In Rule (asgn), $\bigsqcup\{Q_{\sigma[x \mapsto \text{val}_{\sigma}(a)]} \mid \sigma \models Q\}$ denotes the **strongest postcondition** of Q w.r.t. statement $x := a$. It covers all states that are obtained from a state satisfying Q by applying the assignment $x := a$:

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- If $P = \emptyset$ (and thus $\text{Abs}(P) = \{\text{true}, \text{false}\}$) and if no $b \in \text{BExp}_c$ is a tautology or contradiction (i.e., resp. equivalent to **true** or **false**), then the abstract transition system corresponds to the **control flow graph** of c

An Example

Example 15.10

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if [x > y]1 then
  while [¬(y = 0)]2 do
    [x := x - 1;]3;
    [y := y - 1;]4
  end;
  if [x > y]5 then
    [skip]6
  else
    [skip]7;
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else
  [skip]8
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- **Remark:** $\rho_1 := (x > y)$ alone not sufficient to prove loop invariant
(as not necessarily valid after label 3)