Static Program Analysis

Lecture 15: Abstract Interpretation V (Numerical & Predicate Abstraction)

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Overview of Numerical Abstraction Domains

Non-Relational Abstraction Domains

Here, abstract values are independently referring to single variables:

Example 15.1 (Non-relational domains)

- Signs (cf. Example 11.3): $\mathrm{sgn}(x) = s$ ($x \in \mathit{Var}$, $s \in \{+, -, 0\}$)
- Intervals (cf. Example 11.4): $x \in J$ ($x \in \mathit{Var}$, $J \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \cup \{\emptyset\}$)
- Parities (cf. Example 11.2): $x \in \mathbb{Z}_p$ ($x \in \mathit{Var}$, $p \in \{\mathrm{even}, \mathrm{odd}\}$)
- Congruences (cf. Lemma 14.2): $x \bmod m = k$ ($x \in \mathit{Var}$, $m > 1$, $k \in \{0, \ldots, m-1\}$)

Observations

- Expressive power:
  - $\mathrm{Signs} < \mathrm{Intervals}$ (since $+ \cong [1, +\infty]$, ...)
  - $\mathrm{Parities} < \mathrm{Congruences}$ (since x even $\iff$ $x \bmod 2 = 0$, ...)
  - Intervals and Congruences are "incomparable"
- Congruences can prove disequalities ("$x \neq y$") but not inequalities ("$x \leq y$")
  - e.g., $x \bmod m = k_x, y \bmod m = k_y, k_x \neq k_y \implies$ no zero division in $1/(x-y)$
  - but $x \leq y$ generally not representable
- Non-relational domains efficient to represent and manipulate

Relational Abstraction Domains

Here, interdependencies between variables are captured:

Example 15.2 (Relational domains)

- Difference Bound Matrices (DBMs): conjunctions of $x - y \leq c$ and $\pm x \leq c$ ($x, y \in Var$, $c \in \mathbb{Z}$)
- Octagons: conjunctions of $ax + by \leq c$ ($x, y \in Var$, $a, b \in \{-1, 0, 1\}$, $c \in \mathbb{Z}$)
- Octahedra: conjunctions of $a_1 x_1 + \ldots + a_n x_n \leq c$ ($x_i \in Var$, $a_i \in \{-1, 0, 1\}$, $c \in \mathbb{Z}$)
- Polyhedra: conjunctions of $a_1 x_1 + \ldots + a_n x_n \leq c$ ($x_i \in Var$, $a_i \in \mathbb{Z}$, $c \in \mathbb{Z}$)

Observations

- Expressive power:
  - $\mathrm{DBMs} < \mathrm{Octagons} < \mathrm{Octahedra} < \mathrm{Polyhedra}$
  - $\mathrm{Intervals} < \mathrm{DBMs}$ (since $x \in [c_1, c_2] \iff -x \leq -c_1 \wedge x \leq c_2$)
- Can prove inequalities but not (general) disequalities
- Representation and manipulation generally more involved
  - Polyhedra require computation of convex hulls (exponential in $|Var|$)

Combining Non-Relational and Relational Domains

Linear Congruences combine features of Congruences and Polyhedra:

- Given by conjunctions of

$$(a_1x_1 + \ldots + a_nx_n) \bmod m = z$$

  $(x_i \in Var, a_i \in \mathbb{Z}, m > 1, z \in \mathbb{Z})$
- Typical application:

$$2x + 1 \bmod m = k_x, y \bmod m = k_y, k_x \neq k_y \implies \text{no zero division in } 1/(2x + 1 - y)$$

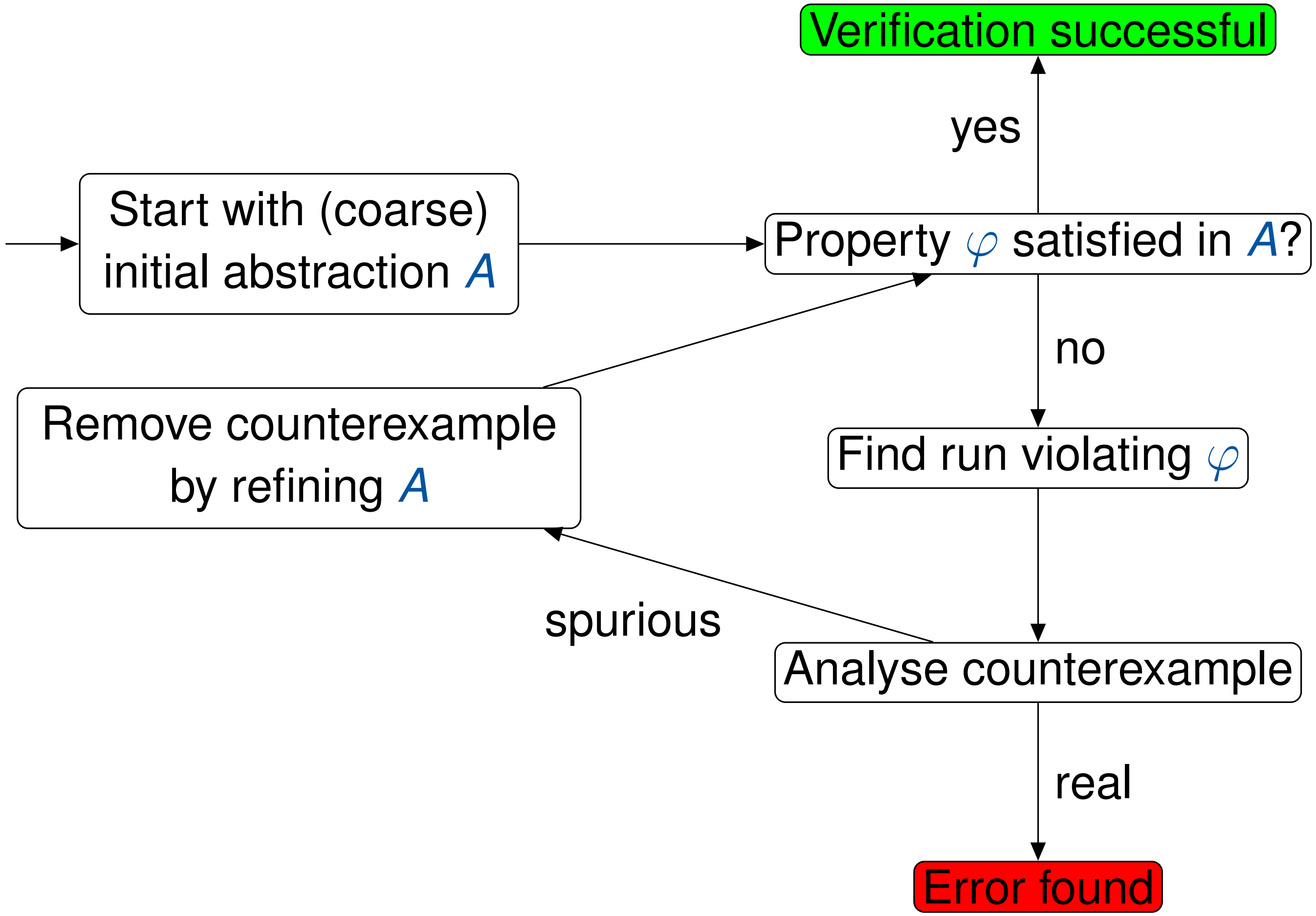
- Again usable for proving disequalities but not inequalities

Overview of Abstraction Refinement Using Predicates

Abstraction Refinement

- **Problem:** desired program property cannot be shown using current abstraction method
- **Reasons:**
  1. program really violates property or
  2. current abstraction is too coarse
- **Solutions:**
  1. fix the problem
  2. refine abstraction
- **Abstraction refinement:** most successful (automatic) method based on
  - predicate abstraction and
  - analysing counterexamples
- **Difference** to standard abstract interpretation with fixed domain:
  abstraction parametrised by and specific to program

Counterexample-Guided Abstraction Refinement (CEGAR)



Overview of Abstraction Refinement Using Predicates

Abstraction Refinement for Predicates

1. Extract predicates (i.e., logical formulae) from counterexample
2. Use Galois connection that classifies program states according to validity of predicates (predicate abstraction)
3. Compute new abstract semantics and search for new counterexamples
4. Iterate until property satisfied or real counterexample found
   (with increasing set of predicates; can entail non-termination)

Predicate Abstraction I

Definition 15.3 (Predicate abstraction)

Let Var be a set of variables.

- A predicate is a Boolean expression $p \in BExp$ over Var.
- A state $\sigma \in \Sigma$ satisfies $p \in BExp$ ($\sigma \models p$) if $val_\sigma(p) = \text{true}$.
- p implies q ($p \models q$) if $\sigma \models q$ whenever $\sigma \models p$ (or: p is stronger than q, q is weaker than p).
- p and q are equivalent ($p \equiv q$) if $p \models q$ and $q \models p$.
- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(P) := \left(\left\{\bigwedge Q \mid Q \subseteq P \cup \neg P\right\}, \models\right).$$

Abbreviations: $\text{true} := \bigwedge \emptyset$, $\text{false} := \bigwedge\{p_i, \neg p_i, \ldots\}$

Predicate Abstraction II

Lemma 15.4

$Abs(P)$ is a complete lattice with

- $\bot = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2}$ *where* $\overline{b} := \bigwedge\{q \in P \cup \neg P \mid b \models q\}$
  (i.e., strongest formula in $Abs(P)$ that is implied by Q_1 and Q_2)
- $Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2}$ *(i.e., strongest formula in $Abs(P)$ that is implied by $Q_1 \vee Q_2$)*

Example 15.5

Let $P := \{p_1, p_2, p_3\}$ with $p_1 := (\mathtt{x} = 1)$, $p_2 := (\mathtt{y} = 2)$, $p_1 := (\mathtt{z} = 3)$.

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain
$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge \neg p_2 \wedge p_3$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} = \neg p_2$$
2. For $Q_1 := p_1 \wedge p_2$ and $Q_2 := p_1 \wedge \neg p_2$, we obtain
$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = \text{false}$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{p_1 \wedge (p_2 \vee \neg p_2)} = p_1$$

Predicate Abstraction III

Important: if predicates are interdependent, then generally

1. $Q_1 \sqcap Q_2 \; (= \overline{Q_1 \wedge Q_2}) \neq \bigwedge(Q_1 \cup Q_2)$ (but $\equiv \bigwedge(Q_1 \cup Q_2)$)
2. $Q_1 \sqcup Q_2 \; (= \overline{Q_1 \vee Q_2}) \neq \bigwedge(Q_1 \cap Q_2)$ (and $\not\equiv \bigwedge(Q_1 \cap Q_2)$)

Example 15.6

1. – $p_1 := (\texttt{x <= y})$, $p_2 := (\texttt{x >= y})$, $p_3 := (\texttt{x = y})$
   – $Q_1 := p_1$, $Q_2 := p_2$
   $\Rightarrow Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge p_2 \wedge p_3 \underset{\equiv}{\neq} \bigwedge(Q_1 \cup Q_2) = p_1 \wedge p_2$
2. – $p_1 := (\texttt{x > y})$, $p_2 := (\texttt{x >= y})$, $p_3 := (\texttt{x = y})$
   – $Q_1 := p_1 \wedge p_2 \wedge \neg p_3$ ($\equiv \texttt{x > y}$), $Q_2 := p_3$
   $\Rightarrow Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} = p_2 \not\equiv \bigwedge(Q_1 \cap Q_2) = \text{true}$

If these dependencies are ignored, then (computationally simpler) Cartesian Abstraction is performed.

Predicate Abstraction IV

Definition 15.7 (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\begin{array}{rlcl} & \alpha : 2^\Sigma \to Abs(P) & \text{and} & \gamma : Abs(P) \to 2^\Sigma \\ \text{with} & \alpha(S) := \bigsqcup\{Q_\sigma \mid \sigma \in S\} & \text{and} & \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\} \\ \text{where} & \multicolumn{3}{l}{Q_\sigma := \bigwedge(\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\})} \end{array}$$

Example 15.8

- Let $Var := \{\mathtt{x}, \mathtt{y}\}$ and $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x} \ \texttt{<=}\ \mathtt{y})$, $p_2 := (\mathtt{x}\ \texttt{=}\ \mathtt{y})$, $p_3 := (\mathtt{x}\ \texttt{>}\ \mathtt{y})$
- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$,
  then $\alpha(S) = Q_{\sigma_1} \sqcup Q_{\sigma_2}$
  $$\begin{aligned} &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3) \\ &= \overline{(p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3)} \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$
- If $Q = p_1 \wedge \neg p_2 \in Abs(P)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

Abstract Semantics for Predicate Abstraction

Definition 15.9 (Execution relation for predicate abstraction)

If $c \in Cmd$ and $Q \in Abs(P)$, then $\langle c, Q\rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\,\frac{}{\langle \texttt{skip}, Q\rangle \Rightarrow \langle \downarrow, Q\rangle} \qquad \text{(asgn)}\,\frac{}{\langle x := a, Q\rangle \Rightarrow \langle \downarrow, \bigsqcup\{Q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models Q\}\rangle}$$

$$\text{(seq1)}\,\frac{\langle c_1, Q\rangle \Rightarrow \langle c_1', Q'\rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, Q\rangle \Rightarrow \langle c_1' ; c_2, Q'\rangle} \qquad \text{(seq2)}\,\frac{\langle c_1, Q\rangle \Rightarrow \langle \downarrow, Q'\rangle}{\langle c_1 ; c_2, Q\rangle \Rightarrow \langle c_2, Q'\rangle}$$

$$\text{(if1)}\,\frac{}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end}, Q\rangle \Rightarrow \langle c_1, \overline{Q \wedge b}\rangle}$$

$$\text{(if2)}\,\frac{}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end}, Q\rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b}\rangle}$$

$$\text{(wh1)}\,\frac{}{\langle \texttt{while } b \texttt{ do } c \texttt{ end}, Q\rangle \Rightarrow \langle c ; \texttt{while } b \texttt{ do } c \texttt{ end}, \overline{Q \wedge b}\rangle}$$

$$\text{(wh2)}\,\frac{}{\langle \texttt{while } b \texttt{ do } c \texttt{ end}, Q\rangle \Rightarrow \langle \downarrow, \overline{Q \wedge \neg b}\rangle}$$

Abstract Semantics for Predicate Abstraction

Remarks

- In Rule (asgn), $\bigsqcup\{Q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models Q\}$ denotes the strongest postcondition of Q w.r.t. statement $x := a$. It covers all states that are obtained from a state satisfying Q by applying the assignment $x := a$:

$$\begin{array}{llcl}
\text{Abstract:} & \langle x := a, Q\rangle & \Rightarrow & \langle\downarrow, \bigsqcup\{Q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models Q\}\rangle \\
& \downarrow\gamma & & \uparrow\alpha \\
\text{Concrete:} & \langle x := a, \{\sigma \in \Sigma \mid \sigma \models Q\}\rangle & \rightarrow & \langle\downarrow, \{\sigma[x \mapsto val_\sigma(a)] \mid \sigma \models Q\}\rangle
\end{array}$$

- In Rules (if1), (if2), (wh1), (wh2), the fact that $b = p_i$ for some $i \in \{1, \ldots, n\}$ implies $Q \wedge [\neg]b \in Abs(P)$, but not $\overline{Q \wedge [\neg]b} \equiv Q \wedge [\neg]b$ (cf. Example 15.6)
  - example: $p_1 := (\mathtt{x > y})$, $p_2 := (\mathtt{x >= y})$, $Q :=$ true, $b := p_1$

  $\Rightarrow \overline{Q \wedge b} = p_1 \wedge p_2 \not\equiv Q \wedge b = p_1$
- An abstract configuration of the form $\langle c, \text{false}\rangle$ represents an unreachable configuration (as there is no $\sigma \in \Sigma$ such that $\sigma \models$ false) and can therefore be omitted
- If $P = \emptyset$ (and thus $Abs(P) = \{\text{true}, \text{false}\}$) and if no $b \in BExp_c$ is a tautology or contradiction (i.e., resp. equivalent to true or false), then the abstract transition system corresponds to the control flow graph of c

An Example

Example 15.10

```
if [x > y]¹ then
  while [¬(y = 0)]² do
    [x := x - 1;]³;
    [y := y - 1;]⁴
  end;
  if [x > y]⁵ then
    [skip]⁶
  else
    [skip]⁷;
  end
else
  [skip]⁸
end
```

- **Claim:** label 7 not reachable (as `x > y` is a loop invariant)
- **Proof:** by predicate abstraction with
  - $p_1 := (\mathtt{x} > \mathtt{y})$
  - $p_2 := (\mathtt{x} >= \mathtt{y})$
- **Abstract transition system:** on the board
- **Remark:** $p_1 := (\mathtt{x} > \mathtt{y})$ alone not sufficient to prove loop invariant (as not necessarily valid after label 3)