Static Program Analysis

Lecture 13: Abstract Interpretation III (Abstract Semantics of WHILE)

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Outline of Lecture 13

Recap: Safe Approximation of Functions and Relations

Systematic Construction of Abstractions

Abstract Semantics of WHILE

Correctness of Abstract Semantics

Recap: Safe Approximation of Functions and Relations

Safe Approximation of Functions I

Definition (Safe approximation)

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $f : L^n \to L$ and $f^\# : M^n \to M$ be functions of rank $n \in \mathbb{N}$. Then $f^\#$ is called a safe approximation of f if, whenever $m_1, \ldots, m_n \in M$,

$$\alpha(f(\gamma(m_1), \ldots, \gamma(m_n))) \sqsubseteq_M f^\#(m_1, \ldots, m_n).$$

Moreover, $f^\#$ is called most precise if the reverse inclusion is also true.

Abstract		**Concrete**
$\vec{m}$	$\xrightarrow{\gamma}$	$\gamma(\vec{m})$
$\downarrow f^\#$		$\downarrow f$
$f^\#(\vec{m}) \sqsupseteq \alpha(f(\gamma(\vec{m})))$	$\xleftarrow{\alpha}$	$f(\gamma(\vec{m}))$

- **Interpretation:** the abstraction $f^\#$ covers all concrete f-results
- **Note:** monotonicity of f and/or $f^\#$ is *not* required (but usually given; see Lemma 12.3)

Encoding Execution Relations by Transition Functions I

- **Reminder:** concrete semantics of WHILE
  - statements $\texttt{skip} \mid x := a \mid c_1 ; c_2 \mid \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end} \mid \texttt{while } b \texttt{ do } c \texttt{ end} \in \mathit{Cmd}$
  - states $\Sigma := \{\sigma \mid \sigma : \mathit{Var} \to \mathbb{Z}\}$ (Definition 11.6)
  - execution relation $\to \subseteq (\mathit{Cmd} \times \Sigma) \times ((\mathit{Cmd} \cup \{\downarrow\}) \times \Sigma)$ (Definition 11.9)
- Yields concrete domain $L := (2^\Sigma, \subseteq)$ and concrete transition function:

Definition (Concrete transition function)

The concrete transition function of WHILE is defined by the family of functions

$$\mathrm{next}_{c,c'} : 2^\Sigma \to 2^\Sigma$$

where $c \in \mathit{Cmd}$, $c' \in \mathit{Cmd} \cup \{\downarrow\}$ and, for every $S \subseteq \Sigma$,

$$\mathrm{next}_{c,c'}(S) := \{\sigma' \in \Sigma \mid \exists \sigma \in S : \langle c, \sigma\rangle \to \langle c', \sigma'\rangle\}.$$

Encoding Execution Relations by Transition Functions II

Remarks: next satisfies the following properties

- "Determinism" (cf. Theorem 11.11):
  - for all $c \in Cmd$, $c' \in Cmd \cup \{\downarrow\}$ and $\sigma \in \Sigma$, $|\text{next}_{c,c'}(\{\sigma\})| \leq 1$
  - for all $c \in Cmd$ and $\sigma \in \Sigma$ there exists exactly one $c' \in Cmd \cup \{\downarrow\}$ such that $\text{next}_{c,c'}(\{\sigma\}) \neq \emptyset$
- When is $\text{next}_{c,c'}(S) = \emptyset$? Possible reasons:
  1. $S = \emptyset$
  2. c' is not a possible successor statement of c, e.g.,
     - $c = (\texttt{x := 0})$
     - $c' = \texttt{skip}$
  3. c' is unreachable for all $\sigma \in S$, e.g.,
     - $c = (\texttt{if x = 0 then x := 1 else skip end})$
     - $c' = \texttt{skip}$
     - $\sigma(\texttt{x}) = 0$ for each $\sigma \in S$

Safe Approximation of Execution Relations

Reminder: abstraction determined by Galois connection (α, γ) with $\alpha : L \to M$, $\gamma : M \to L$

- here: $L := 2^{\Sigma}$, M not fixed
- usually $M = Var \to \ldots$ (more efficient) or $M = 2^{Var \to \cdots}$ (more precise)
- write Abs in place of M
- thus $\alpha : 2^{\Sigma} \to Abs$ and $\gamma : Abs \to 2^{\Sigma}$

Definition (Abstract semantics of WHILE)

Given $\alpha : 2^{\Sigma} \to Abs$, an abstract semantics is defined by a family of functions

$$\mathrm{next}^{\#}_{c,c'} : Abs \to Abs$$

where $c \in Cmd$, $c' \in Cmd \cup \{\downarrow\}$, and each $\mathrm{next}^{\#}_{c,c'}$ is a safe approximation of $\mathrm{next}_{c,c'}$, i.e.,

$$\alpha(\mathrm{next}_{c,c'}(\gamma(abs))) \sqsubseteq_{Abs} \mathrm{next}^{\#}_{c,c'}(abs)$$

for every $abs \in Abs$ (notation: $\langle c, abs\rangle \Rightarrow \langle c', abs'\rangle$ for $\mathrm{next}^{\#}_{c,c'}(abs) = abs'$).

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Systematic Construction of Abstractions

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Correctness of Abstract Semantics

Derivation of Abstract Semantics

- **Problem:** most precise safe approximation not always definable

Example 13.1 (Fermat's Last Theorem)

Sign abstraction (cf. Example 11.3) on

$$\langle \texttt{if n > 2} \wedge \texttt{x^n + y^n = z^n then n := 1 else n := -1 end}, \{[\mathrm{n}, \mathrm{x}, \mathrm{y}, \mathrm{z} \mapsto +]\}\rangle$$

Derivation of Abstract Semantics

- **Problem:** most precise safe approximation not always definable

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- Result $\mathtt{n} \mapsto +$ possible iff there exist $n > 2$ and $x, y, z \geq 1$ such that $x^n + y^n = z^n$
- Thus: most precise approximation yields $\{[\mathtt{x}, \mathtt{y}, \mathtt{z} \mapsto +, \mathtt{n} \mapsto -]\}$ iff equation unsolvable
- Fermat's Last Theorem: equation not solvable
- Final proof by Andrew Wiles and Richard Taylor in 1995

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- More general: solvability of Diophantic equations even undecidable
- Thus: resort to possibly imprecise safe approximations

Systematic Construction of Abstractions

Extraction Functions

- **Assumption:** abstraction determined by pointwise mapping of concrete values
- If $L = 2^C$ and $M = 2^A$ with $\sqsubseteq_L = \sqsubseteq_M = \subseteq$, then $\beta : C \to A$ is called an extraction function
- β determines Galois connection (α, γ) where

$$\begin{aligned}
\alpha &: L \to M : l \mapsto \beta(l) \quad (= \{\beta(c) \mid c \in l\}) \\
\gamma &: M \to L : m \mapsto \beta^{-1}(m) \quad (= \{c \in C \mid \beta(c) \in m\})
\end{aligned}$$

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Example 13.2

1. Parity abstraction (cf. Example 11.2): $\beta : \mathbb{Z} \to \{\text{even}, \text{odd}\}$ where

$$\beta(z) := \begin{cases} \text{even} & \text{if } z \text{ even} \\ \text{odd} & \text{if } z \text{ odd} \end{cases}$$

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2. Sign abstraction (cf. Example 11.3): $\beta : \mathbb{Z} \to \{+, -, 0\}$ with $\beta = \text{sgn}$

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2. Sign abstraction (cf. Example 11.3): $\beta : \mathbb{Z} \to \{+, -, 0\}$ with $\beta = \mathrm{sgn}$
3. Interval abstraction (cf. Example 11.4): not definable by extraction function (as Int is not of the form 2^A)

Safe Approximation by Extraction Functions

Reminder: safe approximation condition (Definition 12.1)

$$\alpha(f(\gamma(m_1), \dots, \gamma(m_n))) \sqsubseteq_M f^{\#}(m_1, \dots, m_n).$$

Safe Approximation by Extraction Functions

Reminder: safe approximation condition (Definition 12.1)

$$\alpha(f(\gamma(m_1), \ldots, \gamma(m_n))) \sqsubseteq_M f^{\#}(m_1, \ldots, m_n).$$

Theorem 13.3

Let $L = 2^C$ and $M = 2^A$ with $\sqsubseteq_L = \sqsubseteq_M = \subseteq$, $\beta : C \to A$ be an extraction function, and $f : C^n \to C$. Then

$$f^{\#} : M^n \to M : (m_1, \ldots, m_n) \mapsto \{\beta(f(c_1, \ldots, c_n)) \mid \forall i \in \{1, \ldots, n\} : c_i \in \beta^{-1}(m_i)\}$$

is a safe approximation of f.

Systematic Construction of Abstractions

Safe Approximation by Extraction Functions

Reminder: safe approximation condition (Definition 12.1)

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is a safe approximation of f.

Proof.

on the board □

Safe Approximation of Arithmetic Operations

Example 13.4 (Sign abstraction)

For $C = \mathbb{Z}$, $A = \{+, -, 0\}$, $\beta = \text{sgn}$:

$+^{\#}$	$\{+\}$	$\{-\}$	$\{0\}$
$\{+\}$	$\{+\}$	$\{+, -, 0\}$	$\{+\}$
$\{-\}$	$\{+, -, 0\}$	$\{-\}$	$\{-\}$
$\{0\}$	$\{+\}$	$\{-\}$	$\{0\}$

$*^{\#}$	$\{+\}$	$\{-\}$	$\{0\}$
$\{+\}$	$\{+\}$	$\{-\}$	$\{0\}$
$\{-\}$	$\{-\}$	$\{+\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

and
$$\begin{aligned}\{+, 0\} *^{\#} \{-\} &= \{+\} *^{\#} \{-\} \cup \{0\} *^{\#} \{-\} \\ &= \{-\} \cup \{0\} \\ &= \{-, 0\}\end{aligned}$$

etc.

Safe Approximation of Boolean Operations

Example 13.5 (Sign abstraction)

1. Relational operations: for $C = \mathbb{Z} \cup \mathbb{B}$, $A = \{+, -, 0\} \cup \mathbb{B}$, $\beta = \mathrm{sgn}$:

$=^\#$	$\{+\}$	$\{-\}$	$\{0\}$
$\{+\}$	$\{\mathrm{true}, \mathrm{false}\}$	$\{\mathrm{false}\}$	$\{\mathrm{false}\}$
$\{-\}$	$\{\mathrm{false}\}$	$\{\mathrm{true}, \mathrm{false}\}$	$\{\mathrm{false}\}$
$\{0\}$	$\{\mathrm{false}\}$	$\{\mathrm{false}\}$	$\{\mathrm{true}\}$

$>^\#$	$\{+\}$	$\{-\}$	$\{0\}$
$\{+\}$	$\{\mathrm{true}, \mathrm{false}\}$	$\{\mathrm{true}\}$	$\{\mathrm{true}\}$
$\{-\}$	$\{\mathrm{false}\}$	$\{\mathrm{true}, \mathrm{false}\}$	$\{\mathrm{false}\}$
$\{0\}$	$\{\mathrm{false}\}$	$\{\mathrm{true}\}$	$\{\mathrm{false}\}$

and $(\{+, 0\} =^\# \{0\}) = (\{+\} =^\# \{0\} \cup \{0\} =^\# \{0\}) = \{\mathrm{false}\} \cup \{\mathrm{true}\} = \{\mathrm{true}, \mathrm{false}\}$ etc.

Safe Approximation of Boolean Operations

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1. Relational operations: for $C = \mathbb{Z} \cup \mathbb{B}$, $A = \{+, -, 0\} \cup \mathbb{B}$, $\beta = \text{sgn}$:

$=^\#$	$\{+\}$	$\{-\}$	$\{0\}$
$\{+\}$	$\{\text{true}, \text{false}\}$	$\{\text{false}\}$	$\{\text{false}\}$
$\{-\}$	$\{\text{false}\}$	$\{\text{true}, \text{false}\}$	$\{\text{false}\}$
$\{0\}$	$\{\text{false}\}$	$\{\text{false}\}$	$\{\text{true}\}$

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   and $(\{+, 0\} =^\# \{0\}) = (\{+\} =^\# \{0\} \cup \{0\} =^\# \{0\}) = \{\text{false}\} \cup \{\text{true}\} = \{\text{true}, \text{false}\}$ etc.

2. Boolean connectives: for $C = A = \mathbb{B}$:

$$\neg^\# = \neg \qquad \wedge^\# = \wedge \qquad \vee^\# = \vee$$

   and
$$\begin{aligned}\{\text{true}, \text{false}\} \wedge^\# \{\text{true}\} &= \{\text{true}\} \wedge^\# \{\text{true}\} \cup \{\text{false}\} \wedge^\# \{\text{true}\} \\ &= \{\text{true}\} \cup \{\text{false}\} \\ &= \{\text{true}, \text{false}\}\end{aligned}$$

   etc.

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Abstract Program States

Now: take values of variables into account

Definition 13.6 (Abstract program state)

Let $\beta : \mathbb{Z} \to A$ be an extraction function.

- An abstract (program) state is an element of the set

$$\{\rho \mid \rho : Var \to A\},$$

called the abstract state space.

Abstract Program States

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- The abstract domain is denoted by $Abs := 2^{Var \to A}$.

Abstract Program States

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called the abstract state space.
- The abstract domain is denoted by $Abs := 2^{Var \to A}$.
- The abstraction function $\alpha : 2^{\Sigma} \to Abs$ is given by
$$\alpha(S) := \{\beta \circ \sigma \mid \sigma \in S\}$$
for every $S \subseteq \Sigma$.

Abstract Evaluation of Expressions

Definition 13.7 (Abstract evaluation functions)

Let $\rho : Var \to A$ be an abstract state.

1. $val^{\#}_{\rho} : AExp \to 2^{A}$ is determined by (f arithmetic operation)

$$val^{\#}_{\rho}(z) := \{\beta(z)\}$$
$$val^{\#}_{\rho}(x) := \{\rho(x)\}$$
$$val^{\#}_{\rho}(f(a_1, \ldots, a_n)) := f^{\#}(val^{\#}_{\rho}(a_1), \ldots, val^{\#}_{\rho}(a_n))$$

Abstract Evaluation of Expressions

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$$\begin{aligned} val^\#_\rho(z) &:= \{\beta(z)\} \\ val^\#_\rho(x) &:= \{\rho(x)\} \\ val^\#_\rho(f(a_1, \ldots, a_n)) &:= f^\#(val^\#_\rho(a_1), \ldots, val^\#_\rho(a_n)) \end{aligned}$$
2. $val^\#_\rho : BExp \to 2^{\mathbb{B}}$ is determined by (g/h relational/Boolean operation)
$$\begin{aligned} val^\#_\rho(t) &:= \{t\} \\ val^\#_\rho(g(a_1, \ldots, a_n)) &:= g^\#(val^\#_\rho(a_1), \ldots, val^\#_\rho(a_n)) \\ val^\#_\rho(h(b_1, \ldots, b_n)) &:= h^\#(val^\#_\rho(b_1), \ldots, val^\#_\rho(b_n)) \end{aligned}$$

Abstract Evaluation of Expressions

Definition 13.7 (Abstract evaluation functions)

Let $\rho : \mathit{Var} \to A$ be an abstract state.

1. $\mathit{val}^{\#}_{\rho} : \mathit{AExp} \to 2^{A}$ is determined by (f arithmetic operation)
$$\begin{aligned} \mathit{val}^{\#}_{\rho}(z) &:= \{\beta(z)\} \\ \mathit{val}^{\#}_{\rho}(x) &:= \{\rho(x)\} \\ \mathit{val}^{\#}_{\rho}(f(a_1,\ldots,a_n)) &:= f^{\#}(\mathit{val}^{\#}_{\rho}(a_1),\ldots,\mathit{val}^{\#}_{\rho}(a_n)) \end{aligned}$$
2. $\mathit{val}^{\#}_{\rho} : \mathit{BExp} \to 2^{\mathbb{B}}$ is determined by (g/h relational/Boolean operation)
$$\begin{aligned} \mathit{val}^{\#}_{\rho}(t) &:= \{t\} \\ \mathit{val}^{\#}_{\rho}(g(a_1,\ldots,a_n)) &:= g^{\#}(\mathit{val}^{\#}_{\rho}(a_1),\ldots,\mathit{val}^{\#}_{\rho}(a_n)) \\ \mathit{val}^{\#}_{\rho}(h(b_1,\ldots,b_n)) &:= h^{\#}(\mathit{val}^{\#}_{\rho}(b_1),\ldots,\mathit{val}^{\#}_{\rho}(b_n)) \end{aligned}$$

Example 13.8 (Sign abstraction)

Let $\rho(\mathtt{x}) = +$ and $\rho(\mathtt{y}) = -$.

1. $\mathit{val}^{\#}_{\rho}(\mathtt{2\ *\ x\ +\ y}) = \{+,-,0\}$
2. $\mathit{val}^{\#}_{\rho}(\neg\mathtt{(x\ +\ 1\ >\ y)}) = \{\text{false}\}$

Abstract Semantics of WHILE I

Reminder: abstract domain is $Abs := 2^{Var \to A}$

Definition 13.9 (Abstract execution relation for statements)

If $c \in Cmd$ and $abs \in Abs$, then $\langle c, abs \rangle$ is called an abstract configuration. The abstract execution relation is defined by the following rules:

$$\text{(skip)}\frac{}{\langle \texttt{skip}, abs \rangle \Rightarrow \langle \downarrow, abs \rangle}$$

$$\text{(asgn)}\frac{}{\langle x := a, abs \rangle \Rightarrow \langle \downarrow, \{\rho[x \mapsto a'] \mid \rho \in abs, a' \in val_\rho^\#(a)\} \rangle}$$

$$\text{(seq1)}\frac{\langle c_1, abs \rangle \Rightarrow \langle c_1', abs' \rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, abs \rangle \Rightarrow \langle c_1' ; c_2, abs' \rangle}$$

$$\text{(seq2)}\frac{\langle c_1, abs \rangle \Rightarrow \langle \downarrow, abs' \rangle}{\langle c_1 ; c_2, abs \rangle \Rightarrow \langle c_2, abs' \rangle}$$

Abstract Semantics of WHILE II

Definition 13.9 (Abstract execution relation for statements; continued)

$$\text{(if1)}\ \frac{\exists\rho \in \mathit{abs} : \mathsf{true} \in \mathit{val}^{\#}_{\rho}(b)}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, \mathit{abs}\rangle \Rightarrow \langle c_1, \mathit{abs} \setminus \{\rho \in \mathit{abs} \mid \mathit{val}^{\#}_{\rho}(b) = \{\mathsf{false}\}\}\rangle}$$

$$\text{(if2)}\ \frac{\exists\rho \in \mathit{abs} : \mathsf{false} \in \mathit{val}^{\#}_{\rho}(b)}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, \mathit{abs}\rangle \Rightarrow \langle c_2, \mathit{abs} \setminus \{\rho \in \mathit{abs} \mid \mathit{val}^{\#}_{\rho}(b) = \{\mathsf{true}\}\}\rangle}$$

$$\text{(wh1)}\ \frac{\exists\rho \in \mathit{abs} : \mathsf{true} \in \mathit{val}^{\#}_{\rho}(b)}{\begin{array}{l}\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \mathit{abs}\rangle \\ \Rightarrow \langle c\texttt{;while}\ b\ \texttt{do}\ c\ \texttt{end}, \mathit{abs} \setminus \{\rho \in \mathit{abs} \mid \mathit{val}^{\#}_{\rho}(b) = \{\mathsf{false}\}\}\rangle\end{array}}$$

$$\text{(wh2)}\ \frac{\exists\rho \in \mathit{abs} : \mathsf{false} \in \mathit{val}^{\#}_{\rho}(b)}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \mathit{abs}\rangle \Rightarrow \langle \downarrow, \mathit{abs} \setminus \{\rho \in \mathit{abs} \mid \mathit{val}^{\#}_{\rho}(b) = \{\mathsf{true}\}\}\rangle}$$

Abstract Semantics of WHILE III

Definition 13.10 (Abstract transition function)

The abstract transition function is defined by the family of mappings

$$\text{next}^{\#}_{c,c'} : Abs \to Abs,$$

given by $\text{next}^{\#}_{c,c'}(abs) := \bigsqcup\{abs' \in Abs \mid \langle c, abs\rangle \Rightarrow \langle c', abs'\rangle\}$

Abstract Semantics of WHILE III

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Example 13.11 (Hailstone Sequences; cf. Example 12.7)

$[\mathtt{skip}]^1;$
$\mathtt{while}\ [\neg(\mathtt{n}\ =\ \mathtt{1})]^2\ \mathtt{do}$
$\quad\mathtt{if}\ [\mathtt{even(n)}]^3\ \mathtt{then}$
$\qquad[\mathtt{n\ :=\ n\ /\ 2}]^4;[\mathtt{skip}]^5$
$\quad\mathtt{else}$
$\qquad[\mathtt{n\ :=\ 3\ *\ n\ +\ 1}]^6;[\mathtt{skip}]^7$
$\quad\mathtt{end}$
$\mathtt{end}$

Execution relation with parity abstraction: see following slide (courtesy B. König)

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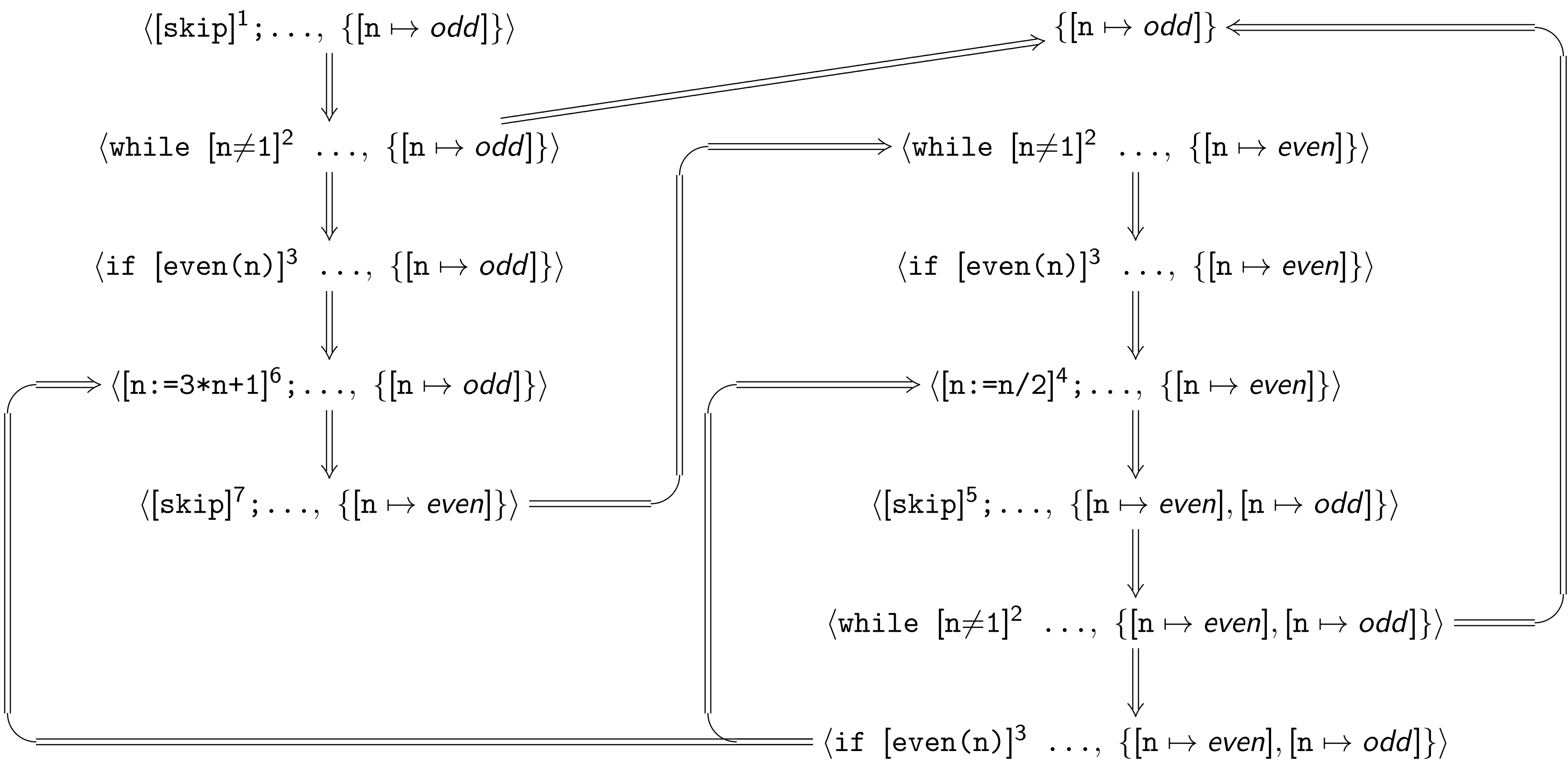
Abstrakte Interpretation von HAILSTONE



Outline of Lecture 13

Correctness of Abstract Semantics

Theorem 13.12 (Soundness of abstract semantics)

For each $c \in Cmd$ *and* $c' \in Cmd \cup \{\downarrow\}$, $\mathrm{next}^{\#}_{c,c'}$ *is a safe approximation of* $\mathrm{next}_{c,c'}$, *i.e., for every* $abs \in Abs$, $\alpha(\mathrm{next}_{c,c'}(\gamma(abs))) \sqsubseteq \mathrm{next}^{\#}_{c,c'}(abs)$.

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For each $c \in Cmd$ and $c' \in Cmd \cup \{\downarrow\}$, $\mathrm{next}^{\#}_{c,c'}$ is a safe approximation of $\mathrm{next}_{c,c'}$, i.e., for every $abs \in Abs$, $\alpha(\mathrm{next}_{c,c'}(\gamma(abs))) \sqsubseteq \mathrm{next}^{\#}_{c,c'}(abs)$.

The soundness proof employs the following auxiliary lemma.

Lemma 13.13 (Soundness of abstract evaluation)

Let $\beta : \mathbb{Z} \to A$ be an extraction function.

1. *For every $a \in AExp$ and $\sigma \in \Sigma$, $\beta(val_\sigma(a)) \in val^{\#}_{\beta\circ\sigma}(a)$.*
2. *For every $b \in BExp$ and $\sigma \in \Sigma$, $val_\sigma(b) \in val^{\#}_{\beta\circ\sigma}(b)$.*

Proof (Lemma 13.13).

omitted □

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For each $c \in Cmd$ *and* $c' \in Cmd \cup \{\downarrow\}$, $\text{next}^{\#}_{c,c'}$ *is a safe approximation of* $\text{next}_{c,c'}$, *i.e., for every* $abs \in Abs$, $\alpha(\text{next}_{c,c'}(\gamma(abs))) \subseteq \text{next}^{\#}_{c,c'}(abs)$.

The soundness proof employs the following auxiliary lemma.

Lemma 13.13 (Soundness of abstract evaluation)

Let $\beta : \mathbb{Z} \to A$ *be an extraction function.*

1. *For every* $a \in AExp$ *and* $\sigma \in \Sigma$, $\beta(val_\sigma(a)) \in val^{\#}_{\beta\circ\sigma}(a)$.
2. *For every* $b \in BExp$ *and* $\sigma \in \Sigma$, $val_\sigma(b) \in val^{\#}_{\beta\circ\sigma}(b)$.

Proof (Lemma 13.13).

omitted □

Proof (Theorem 13.12).

on the board □