Static Program Analysis

Lecture 11: Abstract Interpretation I (Theoretical Foundations)

Winter Semester 2016/17

Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/teaching/ws-1617/spa/

Outline of Lecture 11

Introduction to Abstract Interpretation

Theoretical Foundations of Abstract Interpretation

Excursus: Concrete Semantics of WHILE Programs

Abstract Interpretation I

- **Summary:** a theory of sound approximation of the semantics of programs
- **Basic idea:** execution of program on abstract values
- **Examples:**
  - parity (even/odd) rather than concrete numbers
  - types rather than concrete values (similar to type-level JVM bytecode interpreter)

Abstract Interpretation I

- **Summary:** a theory of sound approximation of the semantics of programs
- **Basic idea:** execution of program on abstract values
- **Examples:**
  - parity (even/odd) rather than concrete numbers
  - types rather than concrete values (similar to type-level JVM bytecode interpreter)
- **Procedure:** run program on (finite) set of abstract values that cover all concrete inputs using abstract operations (for basic operations, statements, ...) that cover all concrete outputs

  $\Longrightarrow$ soundness of approach
- **Preciseness** of information again characterized by partial order

Abstract Interpretation II

- **Advantages:**
  - Abstract interpretation covers conditional branches (`if`/`while`) without further extension
  - Granularity of abstract domain influences precision and complexity of analysis (mutual trade-off)
  - Numerous variants for different kinds of programs (functional, concurrent, ...)
  - Soundness is guaranteed if abstract operations are determined according to theory

Abstract Interpretation II

- **Advantages:**
  - Abstract interpretation covers conditional branches (`if`/`while`) without further extension
  - Granularity of abstract domain influences precision and complexity of analysis (mutual trade-off)
  - Numerous variants for different kinds of programs (functional, concurrent, ...)
  - Soundness is guaranteed if abstract operations are determined according to theory
- **Disadvantages:**
  - Complexity generally higher than with dataflow analysis
  - Automatic derivation of abstract operations can be difficult

Introduction to Abstract Interpretation

Overview

1. Theoretical foundations (Galois connections)
2. (Concrete &) Abstract semantics of WHILE programs
3. Automatic derivation of abstract semantics
4. Application: verification of 16-bit multiplication
5. Predicate abstraction
6. CEGAR (CounterExample-Guided Abstraction Refinement)

Outline of Lecture 11

Introduction to Abstract Interpretation

Theoretical Foundations of Abstract Interpretation

Excursus: Concrete Semantics of WHILE Programs

Galois Connections I

Definition 11.1 (Galois connection)

Let $(L, \sqsubseteq_L)$ and $(M, \sqsubseteq_M)$ be complete lattices. A pair (α, γ) of monotonic functions

$$\alpha : L \to M \quad \text{and} \quad \gamma : M \to L$$

is called a Galois connection if

$$\forall l \in L : l \sqsubseteq_L \gamma(\alpha(l)) \quad \text{and} \quad \forall m \in M : \alpha(\gamma(m)) \sqsubseteq_M m$$

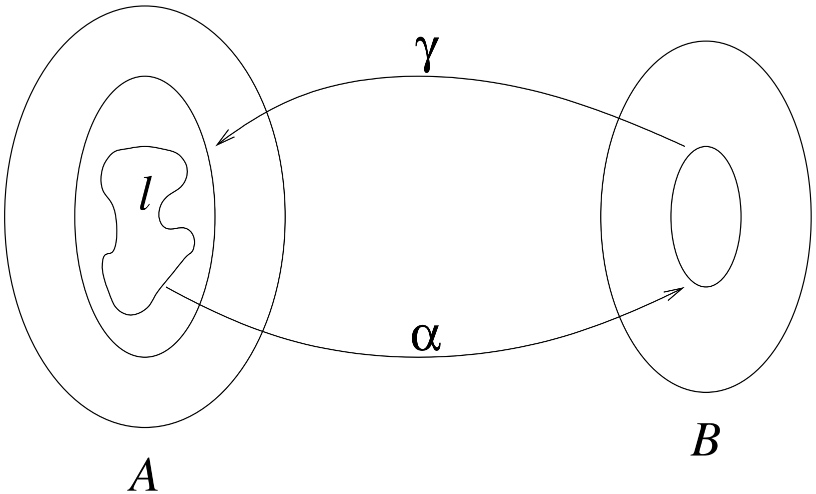
Evariste Galois
(1811–1832)

Interpretation:

- $L = \{\text{sets of concrete values}\}$, $M = \{\text{sets of abstract values}\}$
- $\alpha =$ abstraction function, $\gamma =$ concretisation function
- $l \sqsubseteq_L \gamma(\alpha(l))$: α yields over-approximation
- $\alpha(\gamma(m)) \sqsubseteq_M m$: no loss of precision by abstraction after concretisation
- Usually: $l \neq \gamma(\alpha(l))$, $\alpha(\gamma(m)) = m$

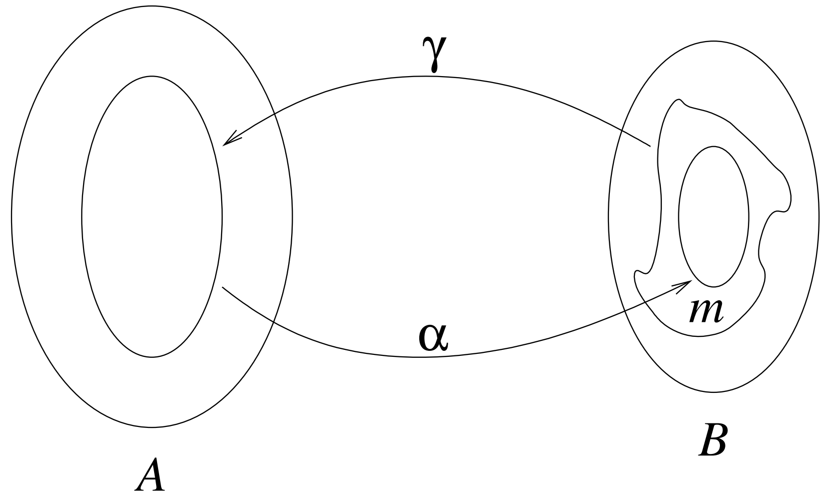
Galois Connections II

For $A = \{\text{concrete values}\}$, $B = \{\text{abstract values}\}$, $L = 2^A$, $M = 2^B$:



$$\forall l \in L : l \sqsubseteq_L \gamma(\alpha(l))$$

(α yields over-approximation)



$$\forall m \in M : \alpha(\gamma(m)) \sqsubseteq_M m$$

(no loss of precision by abstraction after concretisation)

Galois Connections III

Example 11.2 (Parity abstraction)

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

$\gamma : 2^{\{\text{even},\text{odd}\}} \to 2^{\mathbb{Z}}$

$$\gamma(P) := \bigcup_{p \in P} \mathbb{Z}_p$$

where

$$\mathbb{Z}_{\text{even}} := \{\ldots, -2, 0, 2, \ldots\}$$
$$\mathbb{Z}_{\text{odd}} := \{\ldots, -3, -1, 1, 3, \ldots\}$$

Abstract domain

$$M := (2^{\{\text{even},\text{odd}\}}, \subseteq)$$

$\alpha : 2^{\mathbb{Z}} \to 2^{\{\text{even},\text{odd}\}}$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ \{\text{even}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{even}} \\ \{\text{odd}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{odd}} \\ \{\text{even}, \text{odd}\} & \text{otherwise} \end{cases}$$

yields a Galois connection.

Galois Connections III

Example 11.2 (Parity abstraction)

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

$$\gamma : 2^{\{\text{even},\text{odd}\}} \to 2^{\mathbb{Z}}$$

$$\gamma(P) := \bigcup_{p \in P} \mathbb{Z}_p$$

where

$$\mathbb{Z}_{\text{even}} := \{\ldots, -2, 0, 2, \ldots\}$$
$$\mathbb{Z}_{\text{odd}} := \{\ldots, -3, -1, 1, 3, \ldots\}$$

Abstract domain

$$M := (2^{\{\text{even},\text{odd}\}}, \subseteq)$$

$$\alpha : 2^{\mathbb{Z}} \to 2^{\{\text{even},\text{odd}\}}$$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ \{\text{even}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{even}} \\ \{\text{odd}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{odd}} \\ \{\text{even}, \text{odd}\} & \text{otherwise} \end{cases}$$

yields a Galois connection. For example,

- $\gamma(\alpha(\{1,3,7\})) = \gamma(\{\text{odd}\}) = \{\ldots, -3, -1, 1, 3, \ldots\} \supseteq \{1,3,7\}$
- $\alpha(\gamma(\{\text{even}\})) = \alpha(\{\ldots, -2, 0, 2, \ldots\}) = \{\text{even}\}$

Galois Connections IV

Example 11.3 (Sign abstraction)

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

$$\gamma : 2^{\{+,-,0\}} \to 2^{\mathbb{Z}}$$
$$\gamma(S) := \bigcup_{s \in S} \mathbb{Z}_s$$

where

$$\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$$
$$\mathbb{Z}_- := \{-1, -2, -3, \ldots\}$$
$$\mathbb{Z}_0 := \{0\}$$

Abstract domain

$$M := (2^{\{+,-,0\}}, \subseteq)$$

$$\alpha : 2^{\mathbb{Z}} \to 2^{\{+,-,0\}}$$
$$\alpha(Z) := \{\mathrm{sgn}(z) \mid z \in Z\}$$

where

$$\mathrm{sgn}(z) := \begin{cases} + & \text{if } z > 0 \\ - & \text{if } z < 0 \\ 0 & \text{otherwise} \end{cases}$$

yields a Galois connection.

Galois Connections IV

Example 11.3 (Sign abstraction)

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

$$\gamma : 2^{\{+,-,0\}} \to 2^{\mathbb{Z}}$$
$$\gamma(S) := \bigcup_{s \in S} \mathbb{Z}_s$$

where

$$\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$$
$$\mathbb{Z}_- := \{-1, -2, -3, \ldots\}$$
$$\mathbb{Z}_0 := \{0\}$$

Abstract domain

$$M := (2^{\{+,-,0\}}, \subseteq)$$

$$\alpha : 2^{\mathbb{Z}} \to 2^{\{+,-,0\}}$$
$$\alpha(Z) := \{\mathrm{sgn}(z) \mid z \in Z\}$$

where

$$\mathrm{sgn}(z) := \begin{cases} + & \text{if } z > 0 \\ - & \text{if } z < 0 \\ 0 & \text{otherwise} \end{cases}$$

yields a Galois connection. For example,

- $\gamma(\alpha(\{0, 1, 3\})) = \gamma(\{+, 0\}) = \{0, 1, 2, 3, \ldots\} \supseteq \{0, 1, 3\}$
- $\alpha(\gamma(\{+, -\})) = \alpha(\mathbb{Z} \setminus \{0\}) = \{+, -\}$

Galois Connections V

Example 11.4 (Interval abstraction (cf. Slide 7.16))

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

Abstract domain

$$M := (Int, \subseteq)$$

where $Int := (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \cup \{\emptyset\}$

$$\gamma : Int \to 2^{\mathbb{Z}}$$

$$\gamma(J) := \begin{cases} \emptyset & \text{if } J = \emptyset \\ \{z \in \mathbb{Z} \mid z_1 \leq z \leq z_2\} & \\ & \text{if } J = [z_1, z_2] \end{cases}$$

$$\alpha : 2^{\mathbb{Z}} \to Int$$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ [\bigsqcap Z, \bigsqcup Z] & \text{otherwise} \end{cases}$$

yields a Galois connection.

Galois Connections V

Example 11.4 (Interval abstraction (cf. Slide 7.16))

Concrete domain

$$L := (2^{\mathbb{Z}}, \subseteq)$$

Abstract domain

$$M := (\mathit{Int}, \subseteq)$$

where $\mathit{Int} := (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \cup \{\emptyset\}$

$$\gamma : \mathit{Int} \to 2^{\mathbb{Z}}$$

$$\gamma(J) := \begin{cases} \emptyset & \text{if } J = \emptyset \\ \{z \in \mathbb{Z} \mid z_1 \leq z \leq z_2\} & \\ & \text{if } J = [z_1, z_2] \end{cases}$$

$$\alpha : 2^{\mathbb{Z}} \to \mathit{Int}$$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ [\bigsqcap Z, \bigsqcup Z] & \text{otherwise} \end{cases}$$

yields a Galois connection. For example,

- $\gamma(\alpha(\{1, 3, 5, \ldots\})) = \gamma([1, +\infty]) = \{1, 2, 3, 4, 5, \ldots\} \supseteq \{1, 3, 5, \ldots\}$
- $\alpha(\gamma([-1, 1])) = \alpha(\{-1, 0, 1\}) = [-1, 1]$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$
2. γ *is uniquely determined by* α *as follows:* $\gamma(m) = \bigsqcup\{l \in L \mid \alpha(l) \sqsubseteq_M m\}$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$
2. γ *is uniquely determined by* α *as follows:* $\gamma(m) = \bigsqcup\{l \in L \mid \alpha(l) \sqsubseteq_M m\}$
3. α *is uniquely determined by* γ *as follows:* $\alpha(l) = \bigsqcap\{m \in M \mid l \sqsubseteq_L \gamma(m)\}$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) *be a Galois connection with* $\alpha : L \to M$ *and* $\gamma : M \to L$*, and let* $l \in L$*,* $m \in M$*,* $L' \subseteq L$*,* $M' \subseteq M$*.*

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$
2. γ is *uniquely determined by* α *as follows:* $\gamma(m) = \bigsqcup\{l \in L \mid \alpha(l) \sqsubseteq_M m\}$
3. α is *uniquely determined by* γ *as follows:* $\alpha(l) = \bigsqcap\{m \in M \mid l \sqsubseteq_L \gamma(m)\}$
4. α is *completely distributive: for every* $L' \subseteq L$, $\alpha(\bigsqcup L') = \bigsqcup\{\alpha(l) \mid l \in L'\}$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$
2. γ is *uniquely determined by* α *as follows:* $\gamma(m) = \bigsqcup\{l \in L \mid \alpha(l) \sqsubseteq_M m\}$
3. α is *uniquely determined by* γ *as follows:* $\alpha(l) = \bigsqcap\{m \in M \mid l \sqsubseteq_L \gamma(m)\}$
4. α is *completely distributive: for every* $L' \subseteq L$, $\alpha(\bigsqcup L') = \bigsqcup\{\alpha(l) \mid l \in L'\}$
5. γ is *completely multiplicative: for every* $M' \subseteq M$, $\gamma(\bigsqcap M') = \bigsqcap\{\gamma(m) \mid m \in M'\}$

Properties of Galois Connections

Lemma 11.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

1. $\alpha(l) \sqsubseteq_M m \iff l \sqsubseteq_L \gamma(m)$
2. γ is *uniquely determined by* α as follows: $\gamma(m) = \bigsqcup\{l \in L \mid \alpha(l) \sqsubseteq_M m\}$
3. α is *uniquely determined by* γ as follows: $\alpha(l) = \bigsqcap\{m \in M \mid l \sqsubseteq_L \gamma(m)\}$
4. α is *completely distributive*: for every $L' \subseteq L$, $\alpha(\bigsqcup L') = \bigsqcup\{\alpha(l) \mid l \in L'\}$
5. γ is *completely multiplicative*: for every $M' \subseteq M$, $\gamma(\bigsqcap M') = \bigsqcap\{\gamma(m) \mid m \in M'\}$

Proof.

on the board □

Outline of Lecture 11

Introduction to Abstract Interpretation

Theoretical Foundations of Abstract Interpretation

Excursus: Concrete Semantics of WHILE Programs

Excursus: Concrete Semantics of WHILE Programs

Reminder: Syntax of WHILE

The syntax of WHILE Programs is defined by the following context-free grammar (cf. Definition 1.3):

$$a ::= z \mid x \mid a_1\texttt{+}a_2 \mid a_1\texttt{-}a_2 \mid a_1\texttt{*}a_2 \in AExp$$

$$b ::= t \mid a_1\texttt{=}a_2 \mid a_1\texttt{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp$$

$$c ::= \texttt{skip} \mid x \texttt{ := } a \mid c_1 \texttt{;} c_2 \mid \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end} \mid \texttt{while } b \texttt{ do } c \texttt{ end} \in Cmd$$

Excursus: Concrete Semantics of WHILE Programs

Program States

- Meaning of expression = value (in the usual sense)
- Depends on the values of the variables in the expression

Program States

- Meaning of expression = value (in the usual sense)
- Depends on the values of the variables in the expression

Definition 11.6 (Program state)

A (program) state is an element of the set

$$\Sigma := \{\sigma \mid \sigma : \mathit{Var} \to \mathbb{Z}\},$$

called the state space.

Thus $\sigma(x)$ denotes the value of $x \in \mathit{Var}$ in state $\sigma \in \Sigma$.

Evaluation of Expressions

Definition 11.7 (Evaluation function)

Let $\sigma \in \Sigma$ be a state.

1. $val_\sigma : AExp \to \mathbb{Z} : a \to val_\sigma(a)$ yields the value of a in state σ
2. $val_\sigma : BExp \to \mathbb{B} : b \to val_\sigma(b)$ yields the value of b in state σ

Evaluation of Expressions

Definition 11.7 (Evaluation function)

Let $\sigma \in \Sigma$ be a state.

1. $val_\sigma : AExp \to \mathbb{Z} : a \to val_\sigma(a)$ yields the value of a in state σ
2. $val_\sigma : BExp \to \mathbb{B} : b \to val_\sigma(b)$ yields the value of b in state σ

Example 11.8

Let $\sigma(\texttt{x}) = 1$ and $\sigma(\texttt{y}) = 2$.

1. $val_\sigma(\texttt{2 * x + y}) = 4$
2. $val_\sigma(\neg\texttt{(x + 1 > y)}) = \text{true}$

Derivation Rules

- Definition employs derivation rules of the form

$$(\text{Name})\ \frac{\text{Premise(s)}}{\text{Conclusion}}$$

  - meaning: if every premise is fulfilled, then conclusion can be drawn
  - a rule with no premises is called an axiom

Derivation Rules

- Definition employs derivation rules of the form

$$\text{Name}\,\frac{\text{Premise(s)}}{\text{Conclusion}}$$

  – meaning: if every premise is fulfilled, then conclusion can be drawn
  – a rule with no premises is called an axiom
- Iterated application yields complete derivation tree
  – initial program and state at root
  – premises as children of inner nodes
  – axioms at leafs

Execution of Statements I

Definition 11.9 (Execution relation for statements)

If $c \in Cmd$ and $\sigma \in \Sigma$, then $\langle c, \sigma \rangle$ is called a configuration. The execution relation

$$\rightarrow \subseteq (Cmd \times \Sigma) \times ((Cmd \cup \{\downarrow\}) \times \Sigma)$$

is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, \sigma \rangle \rightarrow \langle \downarrow, \sigma \rangle}$$

$$\text{(asgn)}\ \frac{}{\langle x := a, \sigma \rangle \rightarrow \langle \downarrow, \sigma[x \mapsto val_\sigma(a)] \rangle}$$

$$\text{(seq1)}\ \frac{\langle c_1, \sigma \rangle \rightarrow \langle c_1', \sigma' \rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, \sigma \rangle \rightarrow \langle c_1' ; c_2, \sigma' \rangle}$$

$$\text{(seq2)}\ \frac{\langle c_1, \sigma \rangle \rightarrow \langle \downarrow, \sigma' \rangle}{\langle c_1 ; c_2, \sigma \rangle \rightarrow \langle c_2, \sigma' \rangle}$$

Execution of Statements II

Definition 11.9 (Execution relation for statements; continued)

$$\text{(if1)}\ \frac{val_\sigma(b) = \mathsf{true}}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, \sigma\rangle \to \langle c_1, \sigma\rangle}$$

$$\text{(if2)}\ \frac{val_\sigma(b) = \mathsf{false}}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, \sigma\rangle \to \langle c_2, \sigma\rangle}$$

$$\text{(wh1)}\ \frac{val_\sigma(b) = \mathsf{true}}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \sigma\rangle \to \langle c\,\texttt{;}\,\texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \sigma\rangle}$$

$$\text{(wh2)}\ \frac{val_\sigma(b) = \mathsf{false}}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \sigma\rangle \to \langle \downarrow, \sigma\rangle}$$

Remark: $\downarrow$ indicates successful termination of the program

Excursus: Concrete Semantics of WHILE Programs

An Execution Example

Example 11.10

- $\underbrace{\texttt{y := 1; while } \underbrace{\neg\texttt{(x=1)}}_{b} \texttt{ do } \underbrace{\underbrace{\texttt{y := y*x}}_{c_1}\texttt{;}\underbrace{\texttt{x := x-1}}_{c_2}}_{c_0} \texttt{ end}}_{c}$
- Claim: $\langle c, \sigma \rangle \to^+ \langle \downarrow, \sigma_{1,6} \rangle$ for every $\sigma \in \Sigma$ with $\sigma(\mathtt{x}) = 3$
- Notation: $\sigma_{i,j}$ means $\sigma(\mathtt{x}) = i$, $\sigma(\mathtt{y}) = j$
- Derivation: on the board

Determinism Property of Execution Relation

This operational semantics is well defined in the following sense:

Theorem 11.11

The execution relation for statements is deterministic, i.e., whenever $c \in Cmd$, $\sigma \in \Sigma$ *and* $\kappa_1, \kappa_2 \in (Cmd \cup \{\downarrow\}) \times \Sigma$ *such that* $\langle c, \sigma \rangle \to \kappa_1$ *and* $\langle c, \sigma \rangle \to \kappa_2$*, then* $\kappa_1 = \kappa_2$.

Proof.

omitted □

Determinism Property of Execution Relation

This operational semantics is well defined in the following sense:

Theorem 11.11

The execution relation for statements is deterministic, i.e., whenever $c \in Cmd$*,* $\sigma \in \Sigma$ *and* $\kappa_1, \kappa_2 \in (Cmd \cup \{\downarrow\}) \times \Sigma$ *such that* $\langle c, \sigma \rangle \to \kappa_1$ *and* $\langle c, \sigma \rangle \to \kappa_2$*, then* $\kappa_1 = \kappa_2$.

Proof.

omitted □

More on formal semantics of programming languages:
Semantics and Verification of Software in next winter semester