General Remarks

- Please hand in your solutions in groups of 3. Either hand in your solutions at the beginning of the exercise class or put them into the box at the chair.
- If you cannot access the L2P due to registration issues please contact us such that we can add you manually.
- We will publish solutions to the exercises in the L2P. All other material, such as slides and exercise sheets, are distributed on our webpage.
- If you have questions regarding the exercises and/or lecture, feel free to write us an email or visit us at our office.

Exercise 1 (Fixpoints): (2 Points)

Provide all fixpoints and indicate the least fixpoint of the following functions.

a) $f : \{0,1\} \to \{0,1\}$ and $g : \{0,1\} \to \{0,1\}$

$$f(x) := \begin{cases} g(0) & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

$$g(x) := \begin{cases} f(0) & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Provide the least FP w.r.t. $\leq$.

b) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$f(x) = \sin(x) + x$

Provide the least FP w.r.t. $\leq$.

c) $f : 2^{\mathbb{N}_0} \to 2^{\mathbb{N}_0}$ with

$$f(M) := \begin{cases} \{x + y \mid x, y \in M\} & \text{if } M \text{ is finite} \\ \mathbb{N}_0 & \text{otherwise} \end{cases}$$

Provide the least FP w.r.t. $\subseteq$.

d) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^4 - x^2 + x$

Provide the least FP w.r.t. $\leq$.

Exercise 2 (Closed Sets): (2 Points)

Let $(D, \sqsubseteq)$ be a complete lattice satisfying ACC. A set $C \subseteq D$ is *closed* if and only if for each chain $G \subseteq C$, we have $\bigsqcup G \in C$. Further, let $f : D \to D$ be a monotone function.

Prove or disprove: For every closed set $C \subseteq D$ with $f(x) \in C$ for each $x \in C$, we have

a) $\mathit{lfp}(f) \in C$,[1]

b) $f(x) \sqsubseteq x$ implies $\mathit{lfp}(f) \sqsubseteq x$.

[1] $\mathit{lfp}(f)$ denotes the least fixed point of f

Exercise 3 (Pointwise Ordering): (5 Points)

Let $(D, \sqsubseteq)$ be a complete lattice. Further, let $D \to D$ be the set of all (total) functions between elements of D. We lift the order $\sqsubseteq$ to the domain $D \to D$ by pointwise application, i.e., we define $(D \to D, \tilde{\sqsubseteq})$, where $\tilde{\sqsubseteq}$ is given by:

$$f \mathrel{\tilde{\sqsubseteq}} g \text{ if and only if } \forall d \in D \;:\; f(d) \sqsubseteq g(d).$$

a) Show that $(D \to D, \tilde{\sqsubseteq})$ is a partial order.

b) Show that $(D \to D, \tilde{\sqsubseteq})$ is a complete lattice.

c) Now assume that D satisfies ACC. Prove or disprove that fixed points of chains are continuous, i.e.,

$$lfp\left(\tilde{\bigsqcup} \mathcal{F}\right) \quad = \quad \bigsqcup \{lfp(f) \mid f \in \mathcal{F}\}$$

holds for all chains $\mathcal{F} \subseteq (D \to D)$ of monotone functions.

Exercise 4 (Constant Propagation): (3 Points)

Determine the MOP solution of the following program's dataflow system for Constant Propagation Analysis.

```
y := 19;
x := y + 23;
z := 1;
while (y < x) begin
   y := x + z
end
y := x * x
```