General Remarks

- Please hand in your solutions in groups of 3. Either hand in your solutions at the beginning of the exercise class or put them into the box at the chair.
- If you have questions regarding the exercises and/or lecture, feel free to write us an email or visit us at our office.

Exercise 1 (Systematic Construction of Galois Connections): (4 Points)

Instead of designing a new Galois connection for every analysis we would like to perform, it is often preferable to construct suitable Galois connections from existing ones. In this exercise, we establish several rules to support such constructions.
More formally, let $(L_1, \alpha_1, \gamma_1, M_1)$ and $(L_2, \alpha_2, \gamma_2, M_2)$ be Galois connections.

a) Show that $(L_1, \alpha_2 \circ \alpha_1, \gamma_1 \circ \gamma_2, M_2)$ is a Galois connection if $M_1 = L_2$.

b) Let $L = L_1 = L_2$. Show that (L, α, γ, M) is a Galois connection, where

$$\begin{aligned} M &= \{\sqcap\{(m_1', m_2') \mid \varphi(m_1, m_2) = \varphi(m_1', m_2')\} \mid m_1 \in M_1, m_2 \in M_2\} \subseteq M_1 \times M_2 \\ \varphi(m_1, m_2) &= \gamma_1(m_1) \sqcap \gamma_2(m_2), m_1 \in M_1, m_2 \in M_2 \\ \alpha(l) &= (\alpha_1(l), \alpha_2(l)) \\ \gamma(m_1, m_2) &= \gamma_1(m_1) \sqcap \gamma_2(m_2) \end{aligned}$$

Hint: You may assume that $(M, \sqsubseteq_M)$ is a complete lattice.

Exercise 2 (Galois Connections): (6 Points)

Decide for each of the tuples below whether they form a Galois connection or not. Justify your answer.

a)

$$(2^{\mathbb{Z}}, \alpha, \gamma, 2^{\{0,1,2,-1,-2\}})$$

$$\alpha(Z) = \{f(z) \mid z \in Z\}, f(z) = \begin{cases} 2 & \text{if } z \in \mathbb{N}_{even} \setminus \{0\} \\ 1 & \text{if } z \in \mathbb{N}_{odd} \\ -2 & \text{if } z \in -\mathbb{N}_{even} \setminus \{0\} \\ -1 & \text{if } z \in -\mathbb{N}_{odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(P) = \bigcup_{p \in P} \{p \cdot n \mid n \in \mathbb{Z}_{>0}\}$$

b) Let $\mathfrak{A}$ be a minimal DFA over the finite alphabet Σ whose set of states is Q. Further, let $L(\mathfrak{A}, q)$ denote the language accepted by $\mathfrak{A}$ if the set of final states is set to $\{q\}$.

$$\begin{aligned} &(2^{\Sigma^*}, \alpha, \gamma, 2^Q) \\ &\alpha(L) = \{q \in Q \mid \exists w \in L \,.\, w \in L(\mathfrak{A}, q)\} \\ &\gamma(P) = \bigcup_{p \in P} L(\mathfrak{A}, p) \end{aligned}$$

c)

$$(2^{2^{\mathbb{Z}}}, \alpha, \gamma, 2^{\mathbb{Z}}),\ \alpha(M) = \bigcup_{m \in M} m,\ \gamma(P) = \{P\}$$

d)

$$(2^{\mathbb{N}}, \alpha, \gamma, 2^{\mathbb{N}}),\ \alpha(M) = \{n \in \mathbb{N} \setminus M \mid n \in M\},\ \gamma(M) = \{n \in \mathbb{N} \setminus M \mid n \in M\}$$

e)

$$\begin{aligned}&(2^{\mathbb{Z}}, \alpha, \gamma, \mathbb{Z} \cup \{-\infty, \infty\}),\\ &\alpha(Z) = \begin{cases} -\infty & Z = \emptyset \\ max\{Z\} & Z \neq \emptyset \end{cases}\\ &\gamma(p) = \{z \in \mathbb{Z} \mid z \leq p\}\end{aligned}$$