Probabilistic Programming (WS16/17)

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Exercise Sheet 5

General remarks:

- **Due date:** January 20th (before the exercise class). You can either hand your solutions in at the beginning of the exercise class or submit them via email to

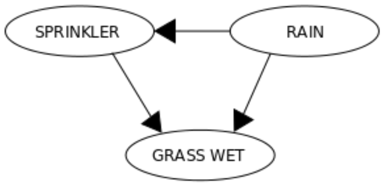
  `propro1617@i2.informatik.rwth-aachen.de`

- Solutions must be written in English.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Bayesian Networks) $\boxed{25\%}$

Suppose that there are two events which could cause grass to be wet: either the sprinkler is on or it's raining. Also, suppose that the rain has a direct effect on the use of the sprinkler (namely that when it rains, the sprinkler is usually not turned on). To model this scenario we use the Bayesian network depicted below.

	SPRINKLER	
RAIN	T	F
F	0.4	0.6
T	0.01	0.99



RAIN	
T	F
0.2	0.8

		GRASS WET	
SPRINKLER	RAIN	T	F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

(a) [15%] Compute the probability that it hadn't rained, given that the grass is dry and the sprinkler is off.

(b) [10%] Translate the Bayesian network into a conditioned `pGCL` program which moreover encodes the evidence (or observation) that the grass is dry and the sprinkler is off.

Exercise 2 (Backward compatibility of transformer cwp) $\boxed{25\%}$

Prove that if c is an unconditioned `pGCL` program and f is an expectation, then

$$\mathsf{cwp}[c](f) \;=\; \mathsf{wp}[c](f)\;.$$

Hint. Use the decomposition lemma of transformer $\underline{\mathsf{cw}}\mathsf{p}$ which says that

$$\underline{\mathsf{cw}}\mathsf{p}[c](f,g) \;=\; \big(\mathsf{wp}[c](f),\, \mathsf{wlp}[c](g)\big)\;.$$

Exercise 3 **(ω–Invariants for Conditional Expectations)** $\boxed{50\%}$

In this exercise, we will analyse the following problem:

Assume you repeatedly flip two fair coins until both turn tails (1). What is the probability that you are finished after *exactly* N trials if in all unsuccessful trials you got at least one tails?

This problem can be modeled as a pGCL program C_{tail} with observe as follows:

$$
\begin{array}{l}
m := 0; \\
c_1, c_2 := 0, 0; \\
\mathsf{while}(c_1 = 0 \vee c_2 = 0)\{ \\
\quad \{c_1 := 0\}\ [0.5]\ \{c_1 := 1\}; \\
\quad \{c_2 := 0\}\ [0.5]\ \{c_2 := 1\}; \\
\quad \mathsf{observe}\ (c_1 = 1 \vee c_2 = 1); \\
\quad m := m + 1 \\
\}
\end{array}
$$

Then, the desired probability is given by $\mathsf{cwp}[C_{tail}]([m = N])$.

Unfortunately, computing this probability relies on a complicated fixed point computation with two variables. As for wp, we would thus like to reason about the $\underline{\mathsf{cwp}}$ of loops $\mathsf{while}\ (G)\ \{C\}$ using invariants. In contrast to our previous proof rule for loops, however, we make use of an ω–invariant, i.e., a sequence of invariants that – in the limit – coincides with the $\underline{\mathsf{cwp}}$ of a loop. More formally, given a loop $\mathsf{while}\ (G)\ \{C\}$, where C is some pGCL program, we define the functional

$$F_{f,g}(X, Y) \;=\; [\neg G] \cdot (f, g) + [G] \cdot \underline{\mathsf{cwp}}[C](X, Y).$$

Then, given a sequence of pairs of expectations and bounded expectations $I_n \in \mathit{Exp} \times \mathit{BExp}$, $n \geq 0$, one can deduce the following proof rule for $\underline{\mathsf{cwp}}$:

$$\frac{F_{f,g}(\mathbf{0}, \mathbf{1}) = I_0 \qquad F_{f,g}(I_n) = I_{n+1}}{\underline{\mathsf{cwp}}[\mathsf{while}\ (G)\ \{C\}](f, g) = \lim_{n \to \infty} I_n}$$

(a) [35%] Show that

$$I_n \;=\; \left([\neg G] \cdot [m = N] + \frac{1}{2}[G] \cdot \sum_{i=1}^{n} \frac{1}{2^i} \cdot [m + i = N]\ ,\ [\neg G] + [G] \cdot \left(\frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2^n} \right) \right)$$

is an ω-invarant of the program C_{tail} (with loop guard $G = (c_1 = 0 \vee c_2 = 0)$) with respect to $([m = N], \mathbf{1})$.

(b) [15%] Use the proof rule for ω-invariants to derive a solution for our problem, i.e., compute $\mathsf{cwp}[C_{tail}]([m = N])$.