Prof. Dr. Ir. Joost-Pieter Katoen
Christoph Matheja   Dr. Federico Olmedo

Lehrstuhl für Informatik 2 Softwaremodellierung und Verifikation
RWTH AACHEN UNIVERSITY

Exercise Sheet 4

General remarks:

- **Due date:** January 13th (before the exercise class). You can either hand your solutions in at the beginning of the exercise class or submit them via email to

  `propro1617@i2.informatik.rwth-aachen.de`

- Solutions must be written in English.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Modelling via Conditional Programs) **30%**

Consider the following problem:

> A box contains three coins: two regular (i.e. fair) coins and one fake two-headed coin (i.e. $\mathsf{Pr}(\mathit{heads}) = 1$). You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

(a) [15%] Write a conditional `pGCL` program to solve this problem.

(b) [15%] State the post–expectation to measure in order to determine the required probability.

Exercise 2 (Well-definedness of Expected Rewards) **20%**

Let $(\mathcal{M}, \mathit{rew})$ be a finite Markov reward model, i.e., $\mathcal{M}$ is a Markov chain with set of states S and $\mathit{rew} : S \to \mathbb{N}$ is a function assigning a reward to every state of $\mathcal{M}$. Moreover, let $s \in S$ be a state in $\mathcal{M}$ and $T \subseteq S$ be a set of target states that is reached from s with probability 1, i.e., $Pr(s \models \lozenge T) = 1$.

Show that the infinite series

$$\mathit{ExpRew}(s \models \lozenge T) \;=\; \sum_{k=0}^{\infty} k \cdot Pr_s\{\pi \in \mathit{Paths}(s) \mid \mathit{rew}(\pi, \lozenge T) = k\}$$

converges.

Here, Paths denotes the set of all paths starting in s. Further, $\mathit{rew}(\pi, \lozenge T)$ is the accumulated reward along the path π until a state from T is reached. If no such state is reached, we set $\mathit{rew}(\pi, \lozenge T) = \emptyset$.

Exercise 3 (From pGCL with Conditioning to Markov Reward Models) **35%**

Consider the following probabilistic program P:

```
a, b, c := 0, 0, 0;
{c := c + 1; } [0.5] {skip};
{a := 1} [0.6] {skip};
if(a = 1){
      {c := c + 1; b := 1} [0.2] {c := c − 1};
} else {
      c := 0
};
observe (a or b)
```

(a) [17.5%] Construct the Markov Reward Model corresponding to P.

(b) [17.5%] Compute the expected value of c after termination of P.

Exercise 4 (Contextual Equivalence under Conditioning) **15%**

Consider the following pair of programs:

$$c_1\textbf{:} \quad x := 1$$
$$c_2\textbf{:} \quad \{x := 1\ [1/2]\ x := 0\};\ \mathsf{observe}\ (x{=}1)$$

They both induce the same probability distribution on variable x (i.e. in both cases x holds value 1 with probability one). Determine whether the following statement is true or false:

For every program context $\mathfrak{C}[\cdot]$, programs $\mathfrak{C}[c_1]$ and $\mathfrak{C}[c_2]$ present always the same behaviour.

Informally, we can understand a *program context* $\mathfrak{C}[\cdot]$ as a program with "holes", and $\mathfrak{C}[c]$ stands for "filling" the holes with c. In case the above statement is true, prove it. In case it is false, provide a program context $\mathfrak{C}[\cdot]$ and an event (i.e. a postcondition) f that distinguishes $\mathfrak{C}[c_1]$ from $\mathfrak{C}[c_2]$ (i.e. the probability that $\mathfrak{C}[c_1]$ establishes f should be different from that of $\mathfrak{C}[c_2]$).