Probabilistic Programming (WS16/17)

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Exercise Sheet 3

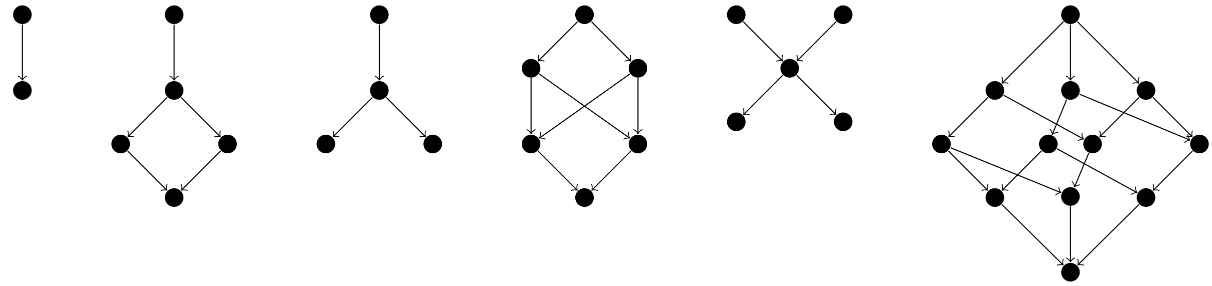
General remarks:

- **Due date:** December 23^{th}.
- Since there is **no exercise class on December 23^{th}**, you may either submit your solution via email to `propro1617@i2.informatik.rwth-aachen.de`, or put your solutions into the box labeled *Static Program Analysis* at the chair. In the latter case, please make sure to mark them as submissions to Probabilistic Programming.
- Solutions must be written in English.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Recapitulation of Complete Lattices) **12%**

One can depict a finite partial order as a graph, where the elements are represented by the nodes and the order relation is given as (the transitive closure of) directed edges (we omit the trivial self-loops at every node). Here, an edge from node u to node v indicates that $v \sqsubseteq u$ and that there is no element that lies 'in between", i.e., there is no $u \neq w \neq v$ such that $v \sqsubseteq w \sqsubseteq u$.

Decide for each of the following graphs whether it represents a complete lattice. Justify your answer.



Exercise 2 (Computing Weakest Preconditions) **38%**

Recall that by the continuity of transformer wp, the weakest pre–expectation of a `pGCL` loop can be given by

$$\mathsf{wp}[\mathsf{while}\ (G)\ \{\mathit{body}\}](f) \;=\; \sup_n F^n(\mathbf{0}) \ ,$$

where $F(X) = [G] \cdot \mathsf{wp}[\mathit{body}](X) + [\neg G] \cdot f$, $\mathbf{0}$ is the constantly 0 expectation and F^n is the composition of F with itself n times, i.e. $F^0 = \mathit{id}$ and $F^{n+1} = F \circ F^n$. Using this formula,

(a) [15%] Give a closed form of

$$\mathsf{wp}\big[\mathsf{while}\ (c = 1)\ \{c := 0\ [1/2]\ c := 1\}\big](f) \ .$$

(b) [15%] Given loop

$$\mathsf{wp}\big[\mathsf{while}\ (c = 1)\ \{c := 0\ [1/2]\ x := x + 1\}\big](x) \ ,$$

you are asked to: First, compute $F^0(\mathbf{0})$, $F^1(\mathbf{0})$, $F^2(\mathbf{0})$ and $F^3(\mathbf{0})$. Second, use these first terms of the sequence to infer the general form of $F^n(\mathbf{0})$.

(c) [8%] Use the result above to compute the expected value of variable x after running program

$$x := 0;\ \mathsf{while}\ (c = 1)\ \{c := 0\ [1/2]\ x := x + 1\}\ .$$

Exercise 3 (On Greatest Fixpoints) **15%**

Let $(D, \sqsubseteq)$ be a complete lattice and $f : D \to D$ be a monotonic function. Show that for every $d \in D$, $d \sqsubseteq f(d)$ implies $d \sqsubseteq \nu f$, where νf denotes the *greatest* fixpoint of f.

Hint. Under the above hypotheses, the greatest fixed point νf can be obtained as $\nu f = \inf_n f^n(\top)$ where $\top$ is the top or greatest element of the lattice.

Exercise 4 (Liberal Probabilistic Invariants) **20%**

In addition to the weakest precondition semantics from the lecture, we can also define a weakest *liberal* precondition (wlp) semantics. The corresponding transformer wlp is defined just like wp except that $\mathsf{wlp}[\mathsf{abort}](f) = \mathbf{1}$ and we take the *greatest* fixpoint for loops instead of the least fixpoint. More formally, we have

$$\mathsf{wlp}[\mathsf{while}\ (G)\ \{C\}](f) = \nu X.\, ([\neg G] \cdot f + [G] \cdot \mathsf{wlp}[C](X))\,.$$

Show that probabilistic loop invariants of wlp imply (in the sense of $\leq$) the wlp of the whole loop w.r.t. postexpectation $[\neg G] \cdot I$. Formally, show that

$$[G] \cdot I \leq \mathsf{wlp}[C](I) \text{ implies } I \leq \mathsf{wlp}[\mathsf{while}\ (G)\ \{C\}]([\neg G] \cdot I).$$

Exercise 5 (Probabilistic Invariants) **15%**

Let $I \leq \mathbf{1}$ be an expectation. Recall that I is a probabilistic invariant of program $P = \mathsf{while}\ (G)\ \{C\}$ whenever $[G] \cdot I \leq \mathsf{wp}[C](I)$.

Show that the probabilistic invariants provided in the lecture are sound for almost surely terminating programs. More formally, show that

$$I \ \leq\ \mathsf{wp}[P]([\neg G] \cdot I) = \mu X\, ([\neg G] \cdot I + [G] \cdot \mathsf{wp}[C](X))\,.$$

whenever I is a probabilistic invariant as above and $\mathsf{wp}[P](\mathbf{1}) = \mathbf{1}$.

Hint: You may assume that we already know for all expectations $f, g \leq \mathbf{1}$ and programs C that

$$\max\{\mathsf{wlp}[C](f) + \mathsf{wp}[C](g) - 1, 0\} \ \leq\ \mathsf{wp}[C](\max\{f + g - 1, 0\}).$$