Exercise Sheet 1

General remarks:

- **Due date:** December 9th (before the exercise class).
- Please submit your solutions via an email to

  `propro1617@i2.informatik.rwth-aachen.de`.

  Your solution should consist of a single file containing your solutions, code, name, and matriculation number.
- Solutions must be written in English.
- While we *might* publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Randomised Quicksort) $\boxed{30\%}$

The randomized Quicksort is a variant of the traditional Quicksort where in each turn the pivot element is chosen at random among all the elements in the array. Its pseudocode looks like

$$
\begin{array}{l}
rQS(A) \triangleq \\
\quad \mathsf{if}\ (|A| \leq 1)\ \mathsf{then\ return}\ (A) \\
\quad pivot := \mathsf{Uniform}(0 ... |A|{-}1) \\
\quad A_< := \{a \in A \mid a < A[pivot]\} \\
\quad A_> := \{a \in A \mid a > A[pivot]\} \\
\quad \mathsf{return}\ \big(rQS(A_<) \mathbin{+\!\!+} A[pivot] \mathbin{+\!\!+} rQS(A_>)\big)
\end{array}
$$

(a) [25%] Implement in WebPPL function rQS. Assume it takes an array of (unrepeated) natural numbers.

(b) [5%] Write the WebPPL code to execute rQS on inputs $[2, 5, 1, 7, 3]$ and $[10, 5, 4, 1]$. Verify that you obtain the expected answer.

Exercise 2 (Random Walk) $\boxed{20\%}$

The function `RW_from_5` below encodes a random walk that starts at position 5, and in each turn moves to the left with probability $p = 0.3$ or to the right with probability $1 - p = 0.7$. The random walk stops when position 0 is reached.

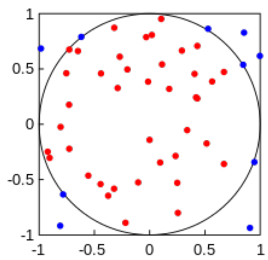
```
var p = 0.3
var RW = function (x)
{
  return (x == 0) ? 0 : (flip(p) ? RW(x+1) : RW(x-1))
}

var RW_from_5 = function() {return RW(5)}
```

Run the program with values $p = 0.1, 0.2, \ldots, 0.9, 1$. What can you empirically deduce from your experiment?

Exercise 3 (Monte Carlo Method) $\boxed{30\%}$

The *Monte Carlo method* allows estimating values through sampling and simulation. A typical application is estimating the value of π. To this end one considers a square enclosing a circle as depicted on the right. Then one samples at random n points within the square and the Monte Carlo method ensures that the relation between the (red) points that fell within the circle and the total (n) number of points approximates the relation between the area of the circle and the area of the square. Symbolically,



$$\frac{N_{\odot}}{N_{\boxdot}} \approx \frac{\mathit{Area}(\bigcirc)}{\mathit{Area}(\square)} ,$$

which in turn gives approximation

$$\pi \approx \frac{4N_{\odot}}{N_{\boxdot}} .$$

(The approximation becomes more precise as one considers a larger number of sampled points). Using this idea, we construct the following program `approx_of_pi` to approximate π:

```
var n = 100
var samples_in_circle = function (m) { ... }
var approx_of_pi = function() {return (4 * samples_in_circle(n)) / n}
// Optional for plotting the distribution
var distr_approx_of_pi = Infer(
  {method: 'rejection', samples: 2000}, approx_of_pi)
viz.auto(distr_approx_of_pi)
//
approx_of_pi()
```

Function `samples_in_circle(m)` samples `m` points at random (i.e. uniformly) in the square and returns the number of these points that fell within the circle.

(a) [25%] Complete the body of function `samples_in_circle`.

(b) [5%] Execute the program 5 times. Write down the 5 values obtained and compute their average.

Exercise 4 (Balls in the Urn) $\boxed{20\%}$

Consider the following puzzle:

> An urn contains 10 black balls and a single red ball. Peter and Paula draw without replacement balls from this urn, alternating after each draw until the red ball is drawn. The game is won by the player who happens to draw the single red ball. Peter is a gentleman and offers Paula the choice of whether she wants to start or not. Paula has a hunch that she might be better off if she starts; after all, she might succeed in the first draw. On the other hand, if her

first draw yields a black ball, then Peter's chances to draw the red ball in his first draw are increased, because then one black ball is already removed from the urn. How should Paula decide in order to maximize her probability of winning?

(a) [15%] Write a program that simulates the game played by Peter and Paula.

(b) [5%] Use the program above to provide approximate values for the probability that Paula wins both when Peter starts and when herself starts. (Include the code that you used to determine these probabilities.)