Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

- bisimulation
- CTL, CTL*-equivalence
- computing the bisimulation quotient ⟵
- abstraction stutter steps
- simulation relations

Bisimulation quotient $\mathcal{T}/\sim$

$\mathcal{T}/\sim$ arises by collapsing all bisimilar states in $\mathcal{T}$

- *states* of $\mathcal{T}/\sim$: bisimulation equivalence classes of $\mathcal{T}$
- *transitions:* arise by lifting $\mathcal{T}$'s transitions to the bisimulation equivalence classes

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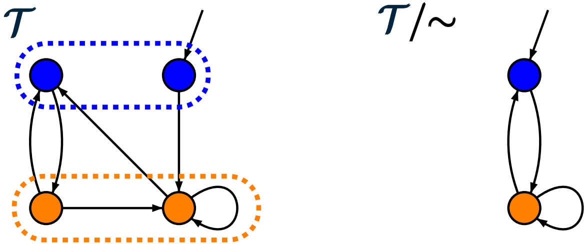

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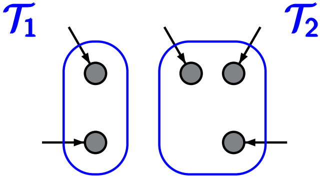

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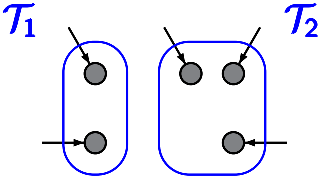

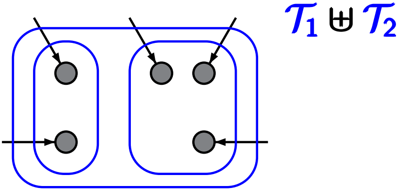

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   regard $\mathcal{T}_1 \uplus \mathcal{T}_2$ and check whether for all bisimulation equivalence classes C in $\mathcal{T}_1 \uplus \mathcal{T}_2$:

$$C \cap S_{0,1} \neq \varnothing \quad \text{iff} \quad C \cap S_{0,2} \neq \varnothing$$

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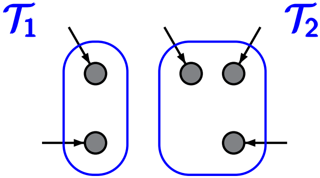

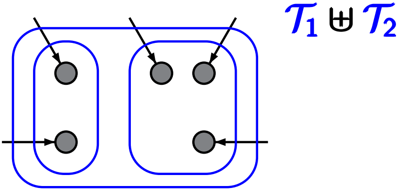

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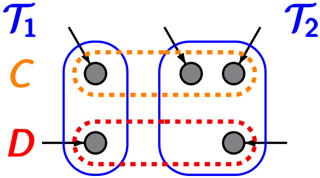

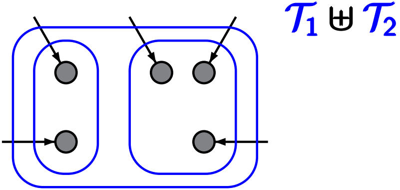

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2. graph minimization:

   replace $\mathcal{T}$ with $\mathcal{T}/\!\sim$ and analyze $\mathcal{T}/\!\sim$

Computing the bisimulation quotient

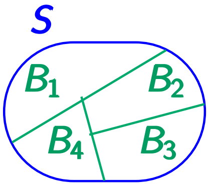
.... relies on a partitioning refinement algorithm ...

here: only explanations for finite transition systems, possibly with terminal states

Notations: partitions, block, superblock

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ finite transition system

partition for $\mathcal{T}$: decomposition of the state space S into pairwise disjoint nonempty subsets



$\mathcal{B} = \{B_1, \ldots, B_k\}$ s.t.

- $B_i \neq \varnothing$
- $B_i \cap B_j = \varnothing$ for $i \neq j$
- $S = B_1 \cup \ldots \cup B_k$

The B_i's are called blocks of $\mathcal{B}$.
A superblock denotes any union of blocks.

Partitions and equivalences

$$\boxed{\text{partitions} \quad \widehat{=} \quad \text{equivalences on } S}$$

- partition $\mathcal{B} \rightsquigarrow$ equivalence relation $\mathcal{R}_{\mathcal{B}}$ where

$$\mathcal{R}_{\mathcal{B}} = \{(s, s') : [s]_{\mathcal{B}} = [s']_{\mathcal{B}}\}$$

$$[s]_{\mathcal{B}} = \text{unique block } B_i \in \mathcal{B} \text{ with } s \in B_i$$

- equivalence $\mathcal{R}$ on $S \rightsquigarrow$ partition $\mathcal{B} = S/\mathcal{R}$

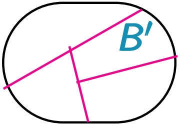
Notations for partitions: finer, coarser

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be partitions for $\mathcal{T}$.

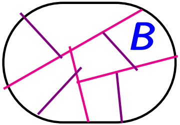
$\mathcal{B}_1$ is called *finer* than $\mathcal{B}_2$ (and $\mathcal{B}_2$ *coarser* than $\mathcal{B}_1$) if

$$\forall B \in \mathcal{B}_1 \ \exists B' \in \mathcal{B}_2 \text{ such that } B \subseteq B',$$

i.e., if all blocks $B' \in \mathcal{B}_2$ are superblocks of $\mathcal{B}_1$



$\mathcal{B}_2$



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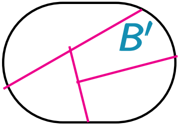

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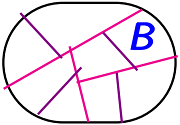

$\mathcal{B}_1$

Example: if $\mathcal{R}$ is a bisimulation for $\mathcal{T}$ and an equivalence then $S/\mathcal{R}$ is *finer* than $S/\sim$

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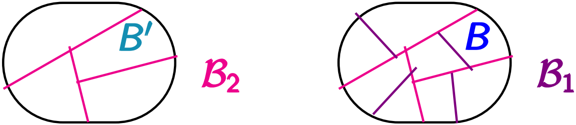
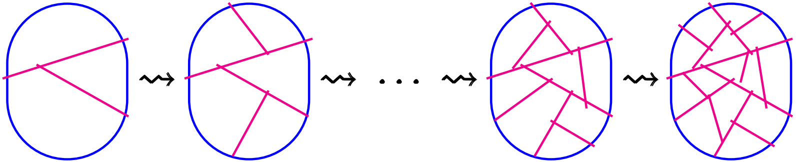


$\mathcal{B}_1$ is called *strictly finer* than $\mathcal{B}_2$ if

(1) $\mathcal{B}_1$ is finer than $\mathcal{B}_2$ and (2) $\mathcal{B}_1 \neq \mathcal{B}_2$

Computing the bisimulation quotient

by stepwise refinement of partitions the state set S



$\mathcal{B}_{AP} = \mathcal{B}_0 \quad \mathcal{B}_1 \quad \ldots \quad \mathcal{B}_{k-1} \quad \mathcal{B}_k = S/\sim$

initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0 = S/\mathcal{R}_{AP}$ where

$$\mathcal{R}_{AP} = \{ (s_1, s_2) : L(s_1) = L(s_2) \}$$

Characterization of $S/\!\sim_{\mathcal{T}}$

... as the coarsest partition of the
state space S such that ....

Bisimulation equivalence $\sim_{\mathcal{T}}$

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2. $\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \\ s_1' & & \end{array}$ can be completed to $\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \downarrow \\ s_1' & \sim_{\mathcal{T}} & s_2' \end{array}$

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where $Pre(C) = \{ s \in S : \exists s' \in C \text{ s.t. } s \to s' \}$

Partitioning refinement algorithm

input: finite TS $\mathcal{T}$ with state space S over AP (possibly with terminal states)

output: bisimulation quotient $S/\sim_{\mathcal{T}}$

Partitioning refinement algorithm

$\mathcal{B}_0 := \mathcal{B}_{AP}$ ← identifies states with the same labeling

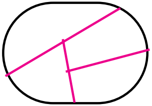
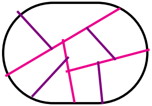
$i := 0$

REPEAT $\mathcal{B}_{i+1} := \mathit{Refine}(\mathcal{B}_i)$

$\quad\quad i := i+1$

UNTIL $\mathcal{B}_i = \mathcal{B}_{i-1}$ ← no more refinement possible
hence: $\mathcal{B}_i = S/\!\sim_{\mathcal{T}}$

return $\mathcal{B}_i$

$\mathcal{B}_i$  $\mathcal{B}_{i+1}$ 

loop invariant:

$\mathcal{B}_i$ is coarser than $S/\!\sim_{\mathcal{T}}$ and finer than $\mathcal{B}_{AP}$

Maximal number of iterations?

$\mathcal{B}_0 := \mathcal{B}_{AP}$; $i := 0$

```
REPEAT
```

$\mathcal{B}_{i+1} := \mathit{Refine}(\mathcal{B}_i)$; $i := i+1$

`UNTIL` no further refinement is possible

Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps **?**

Maximal number of iterations?


```
𝓑₀ := 𝓑_AP;  i := 0
REPEAT
      𝓑_{i+1} := Refine(𝓑_i);  i := i+1
UNTIL  no further refinement is possible
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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps **?**

answer: $|S| - 1$

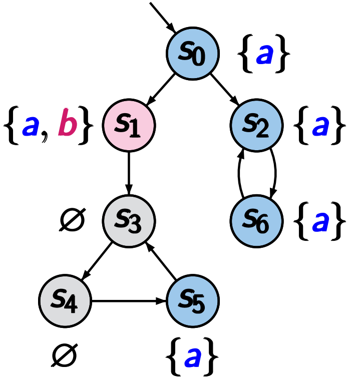
Note that $|\mathcal{B}_i| \geq i+1$.

Hence: if there are $k = |S| - 1$ iterations then $\mathcal{B}_k$ consists of singletons

The initial partition

initial partition $\mathcal{B}_{AP}$:

identifies all states s, t s.t. $L(s) = L(t)$



$$\mathcal{B}_{AP} = \Big\{ \{s_0, s_2, s_6, s_5\}, \{s_1\}, \{s_3, s_4\} \Big\}$$

Initial partition

initial partition $\mathcal{B}_{AP}$:

- identifies all states with the same labeling
- agrees with the quotient under the equivalence

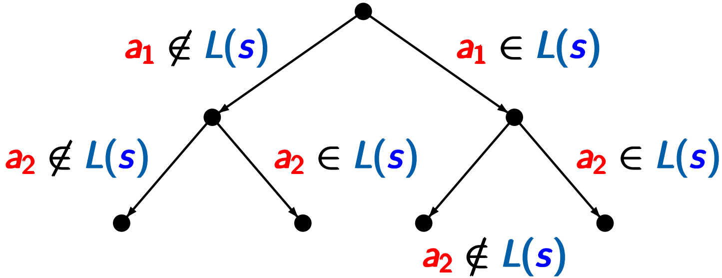
$$s \equiv_{AP} t \quad \text{iff} \quad L(s) = L(t)$$

> compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for AP

compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, \ldots, a_k\}$

inner nodes at level i: decision "$a_i \in L(s)$?"

leaves: sets of states with the same labeling



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compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, ..., a_k\}$

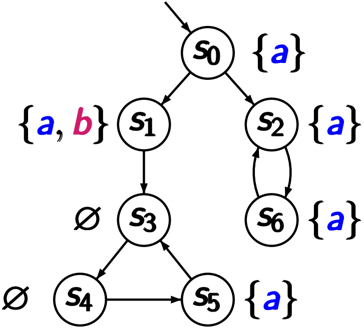
initally: each leaf represents the empty state-set

for each state s:

traverse the decision tree from the root to a leaf v

insert s in the set for v

Example: initial partition



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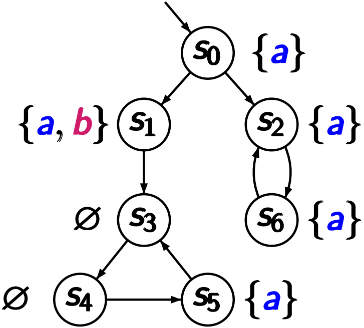

decision tree for
$AP = \{a, b\}$

1. level: $a \in L(s)$?
2. level: $b \in L(s)$?

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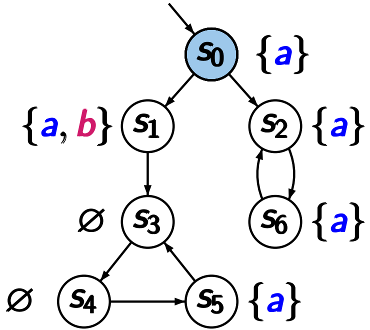

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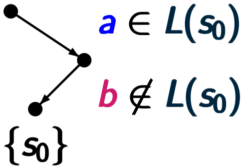

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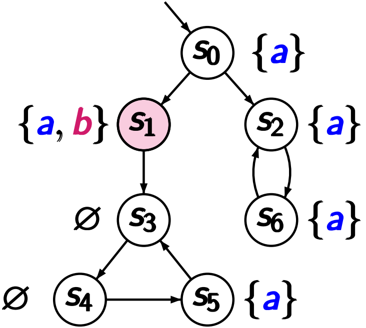

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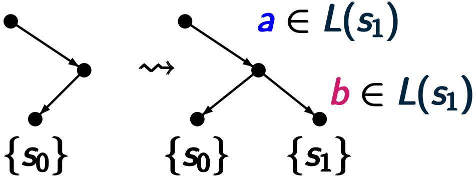

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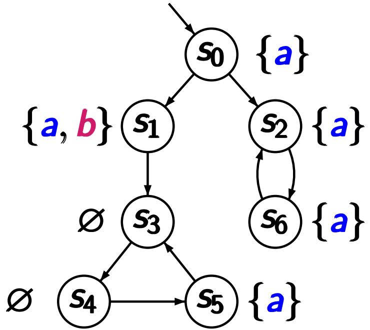

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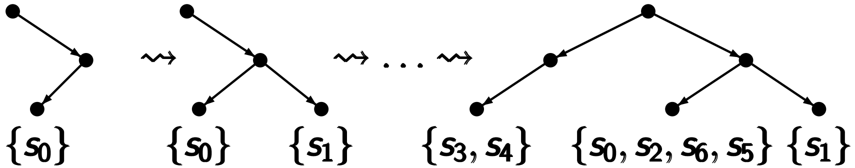

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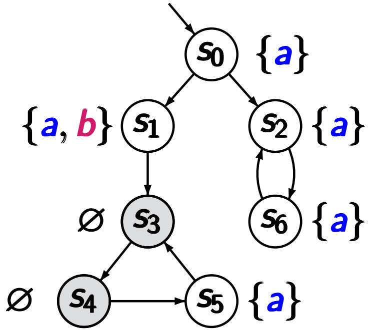

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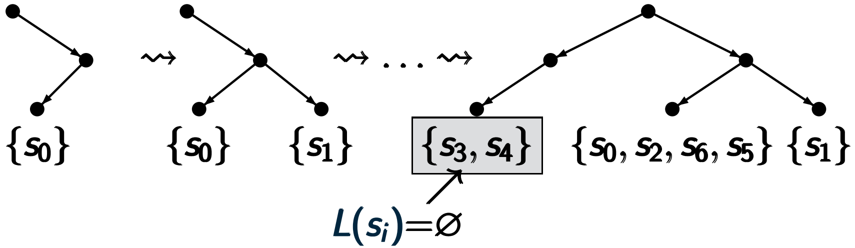

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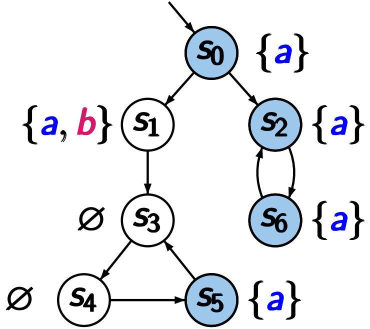

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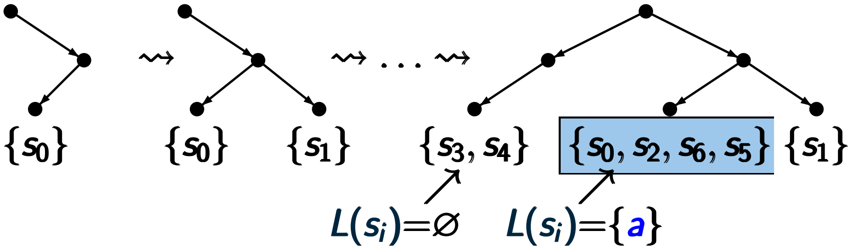

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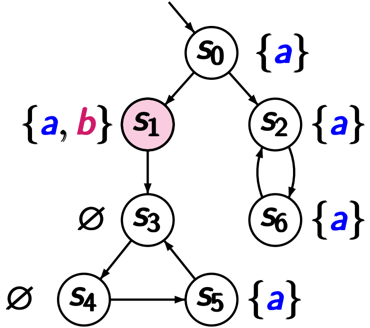
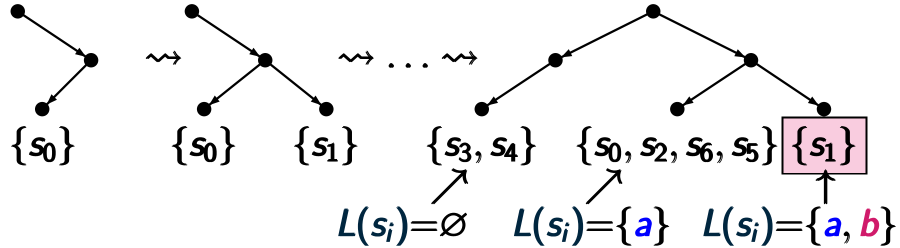


decision tree for
$AP = \{a, b\}$

1. level: $a \in L(s)$?
2. level: $b \in L(s)$?



Computation of the initial partition

```
generate the root node v_0 of the decision tree
FOR ALL states s DO
   v := v_0
   FOR i = 1,...,k OD
      IF a_i ∈ L(s)
         THEN   v := find_or_add(right son of v)
         ELSE   v := find_or_add(left son of v)
      FI
   OD  ←—  v is a leaf of depth k
   add s into the state-set of v
OD
```

suppose $AP = \{a_1, \ldots, a_k\}$

The state-sets of the leaves are the blocks in $\mathcal{B}_{AP}$.

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complexity: $\mathcal{O}(|S| \cdot |AP|)$

The state-sets of the leaves are the blocks in $\mathcal{B}_{AP}$.

Partitioning refinement (schema)

$\mathcal{B} := \mathcal{B}_{AP}$

```
WHILE refinements are possible DO
        B := Refine(B)
OD

return B
```

Partitioning refinement (schema)

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refinement: stabilization for some superblock C of $\mathcal{B}$:

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split each block $B \in \mathcal{B}$ into two blocks:

$B \cap \mathit{Pre}(C)$ and $B \setminus \mathit{Pre}(C)$

Partitioning refinement (schema)

$\mathcal{B} := \mathcal{B}_{AP}$

WHILE refinements are possible DO

$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C) \text{ for some splitter } C}$

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split each block $B \in \mathcal{B}$ into two blocks:

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Partitioning splitter algorithm


```
𝓑 := 𝓑_AP
WHILE  refinements are possible DO
      choose some superblock C of 𝓑;
      𝓑 := Refine(𝓑, C)
OD
return 𝓑
```

Partitioning splitter algorithm

$\mathcal{B} := \mathcal{B}_{AP}$

WHILE refinements are possible DO

  choose some superblock C of $\mathcal{B}$;

  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} \mathit{Refine}(B, C)$

OD

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Partitioning splitter algorithm

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OD

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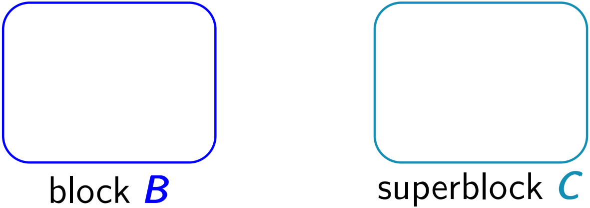
$Refine(B, C)$



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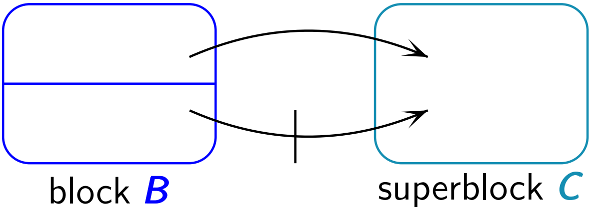

Partitioning splitter algorithm

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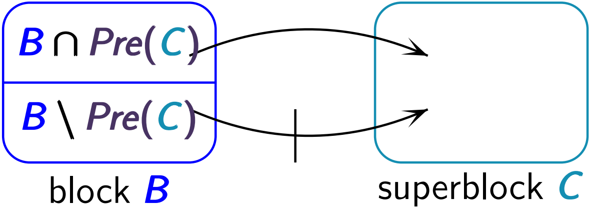

Partitioning splitter algorithm

$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` refinements are possible `DO`

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`OD`

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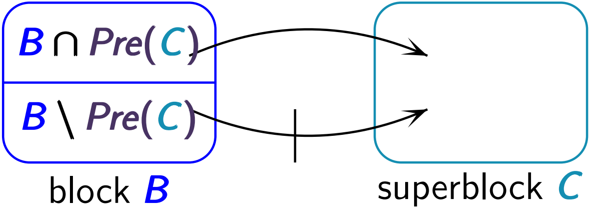

The refinement operator

Let $\mathcal{B}$ be a partition for S and C a superblock of $\mathcal{B}$.

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

where $Refine(B, C) = \{B \cap Pre(C), B \setminus Pre(C)\} \setminus \{\varnothing\}$

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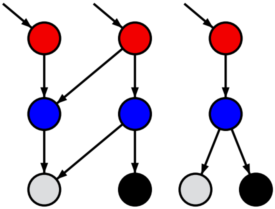
If $\mathcal{B}$ is finer than $\mathcal{B}_{AP}$ and coarser than $S/\!\sim_{\mathcal{T}}$ then:

(a) $Refine(\mathcal{B}, C)$ is finer than $\mathcal{B}$ and $\mathcal{B}_{AP}$

(b) $Refine(\mathcal{B}, C)$ is coarser than $S/\!\sim_{\mathcal{T}}$

(c) $Refine(\mathcal{B}, C) = \mathcal{B}$ for all $C \in \mathcal{B}$ iff $\mathcal{B} = S/\!\sim_{\mathcal{T}}$

Example: partitioning splitter algorithm



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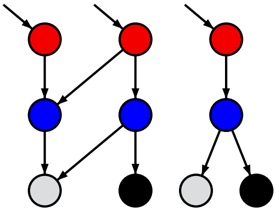
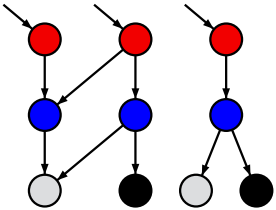


$\Longrightarrow$

refinement
w.r.t. $\bullet$

Example: partitioning splitter algorithm



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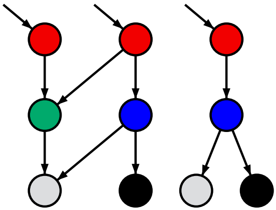
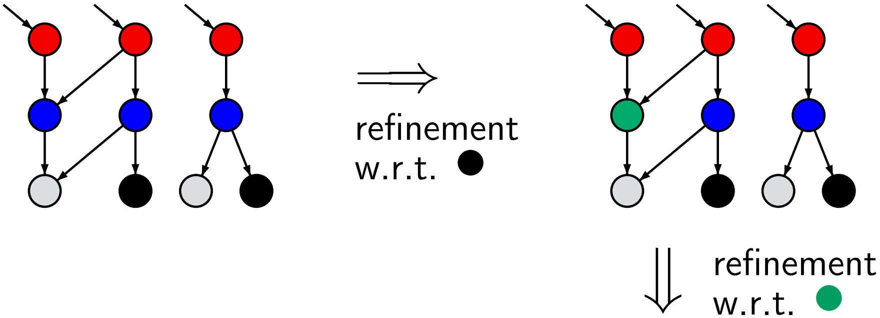


Example: partitioning splitter algorithm



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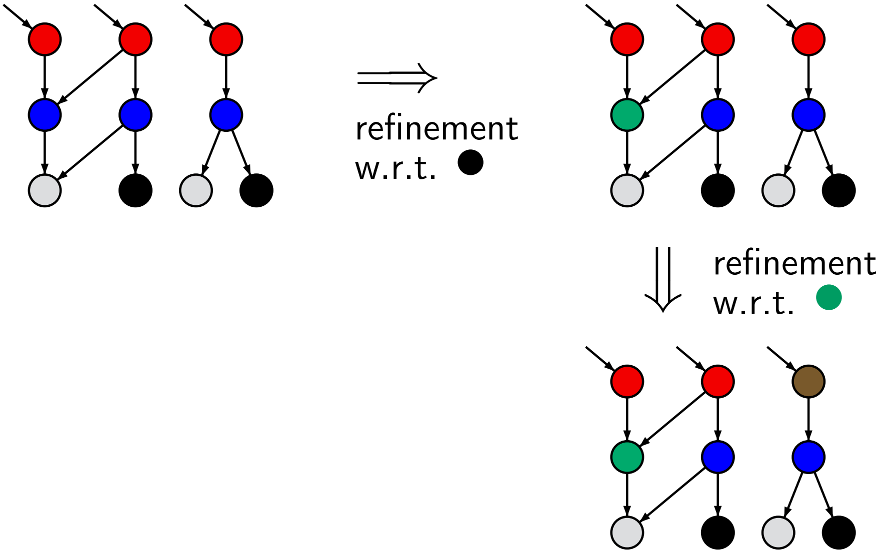

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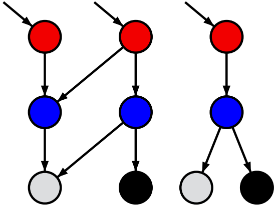

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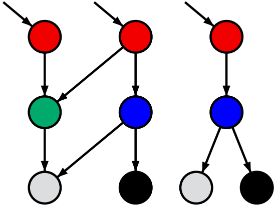
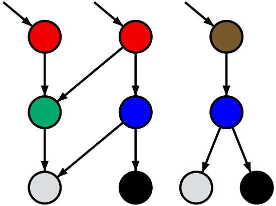


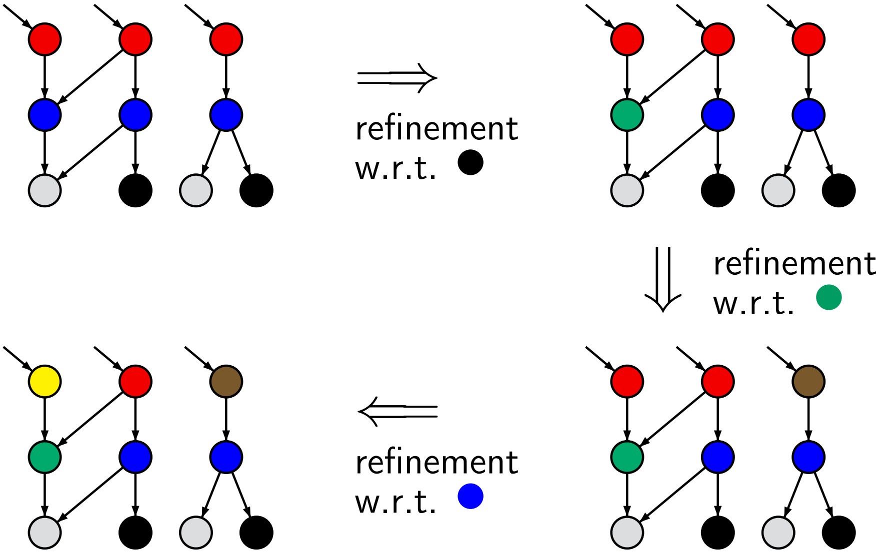
$\Downarrow$ refinement
w.r.t.



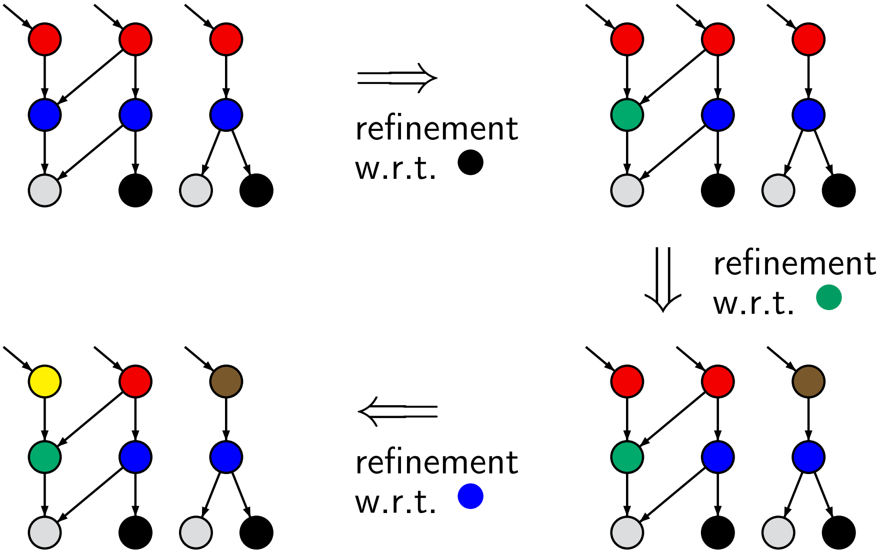
$\Longleftarrow$

refinement
w.r.t.

Example: partitioning splitter algorithm



Example: partitioning splitter algorithm



7 bisimulation equivalence classes

The refinement operator

given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$,
how to compute

$$\mathit{Refine}(\mathcal{B}, C)$$

efficiently **?**

The refinement operator

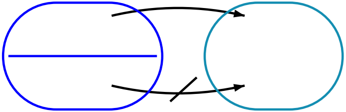
given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$,
how to compute

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

where for all blocks $B \in \mathcal{B}$:

$$Refine(B, C) = \{\, B \cap Pre(C),\ B \setminus Pre(C) \,\} \setminus \{\varnothing\}$$



block B superblock C

Refinement operator *Refine*($\mathcal{B}$, C)

Refinement operator $Refine(\mathcal{B}, C)$


```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO
        "move" state s from block [s]_B = B
               to the new block B ∩ Pre(C)
   OD
OD
```

Refinement operator $\mathit{Refine}(\mathcal{B}, C)$

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... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus \mathit{Pre}(C)$

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OD
```
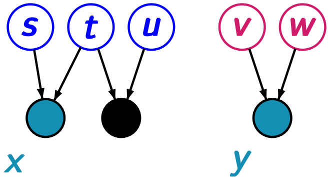

... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus Pre(C)$

time complexity:

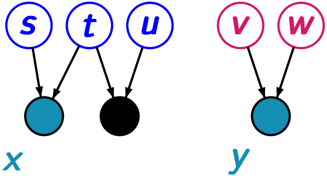
$$\mathcal{O}\Big(\sum_{s' \in C} |Pre(s')| + |C| \Big)$$

Example: refinement operator



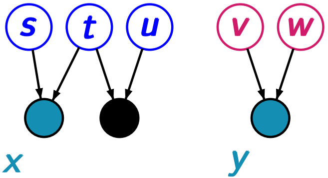
partition $\mathcal{B}$

Example: refinement operator



superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$



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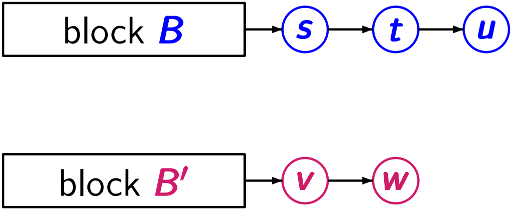
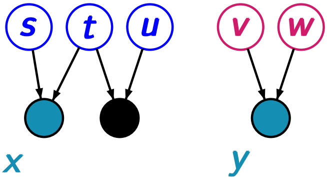


...



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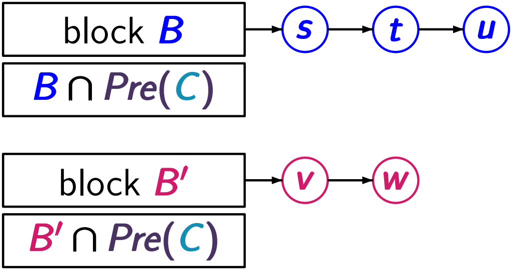
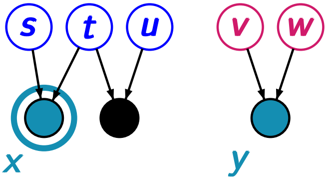


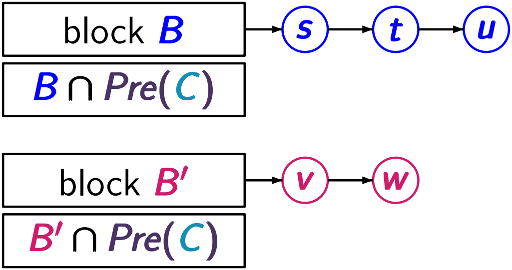
...

Example: refinement operator



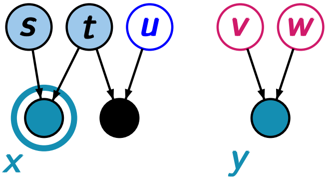
superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow Refine(\mathcal{B}, C)$



...

Example: refinement operator



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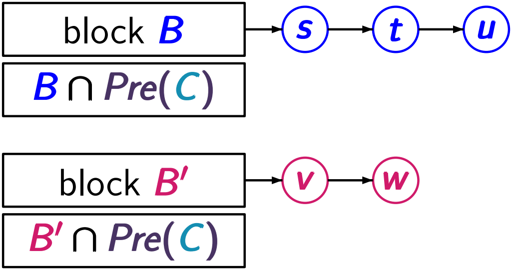
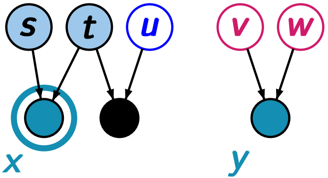


...

Example: refinement operator



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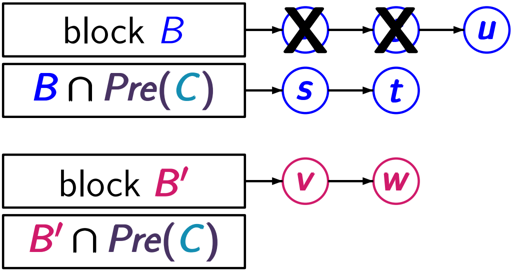
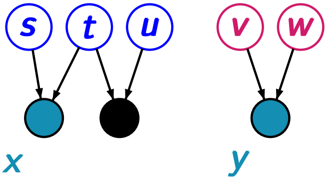


...

Example: refinement operator



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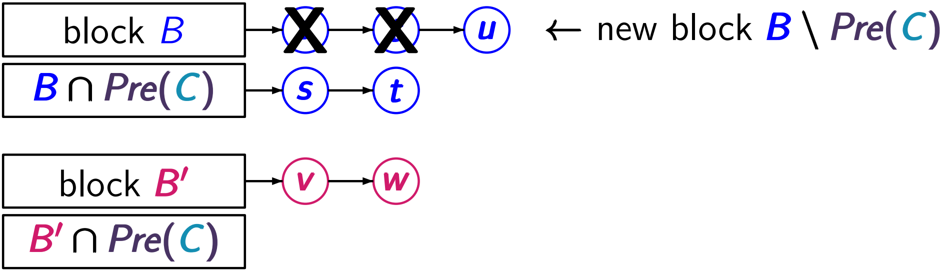
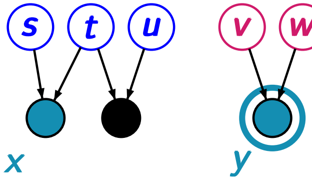


$\ldots$

Example: refinement operator



superblock $C = \{x, y\}$

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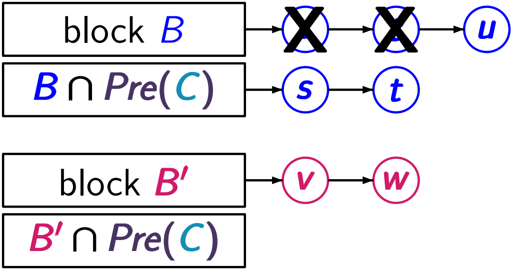
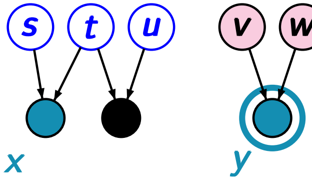


$\leftarrow$ new block $B \setminus Pre(C)$

...

Example: refinement operator



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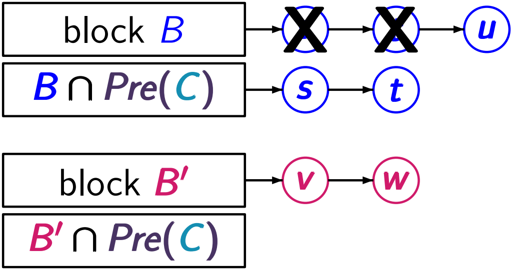

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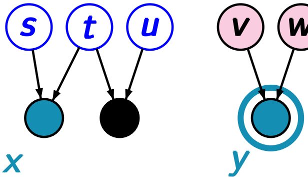

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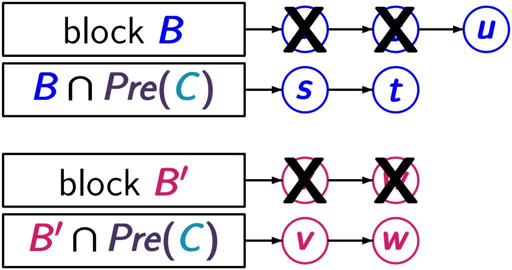

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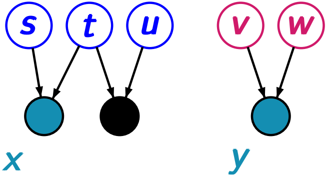

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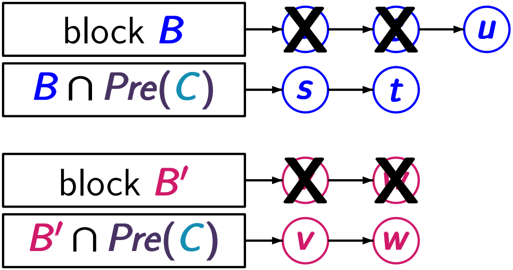
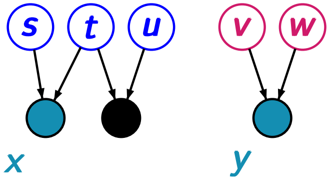


$\leftarrow$ new block $B \setminus Pre(C)$

$\leftarrow B' \setminus Pre(C) = \varnothing$

...

Example: refinement operator



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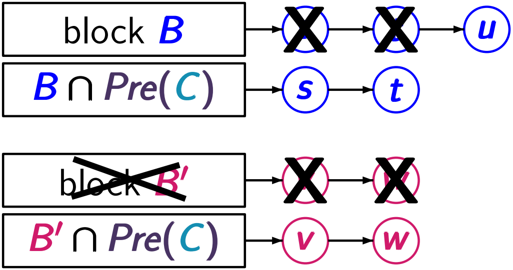

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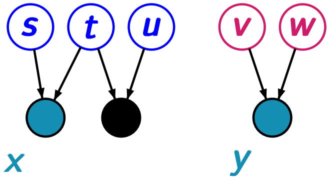
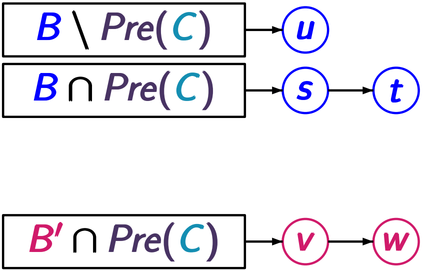


superblock $C = \{x, y\}$

$Refine(\mathcal{B}, C)$



. . .

Partitioning splitter algorithm

$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` there is a splitter C for $\mathcal{B}$ `DO`

    select such a splitter C;
    $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

`OD`

return $\mathcal{B}$

$\mathcal{B} := \mathcal{B}_{AP}$ $\longleftarrow$ $\boxed{\text{time complexity: } \mathcal{O}(|S| \cdot |AP|)}$

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Partitioning splitter algorithm

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time complexity:

$$\mathcal{O}\Big(\sum_{C} \big(\sum_{s' \in C} |\mathit{Pre}(s')| + |C| \big) \; + \; |S| \cdot |AP| \Big)$$

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time complexity:

$$\mathcal{O}\Big(\sum_{C} \big(\sum_{s' \in C} |\mathit{Pre}(s')| + |C| \big) \; + \; |S| \cdot |AP| \Big)$$

+ cost for splitter search and management

Partitioning splitter algorithms

2 instances of the partitioning splitter algorithm that differ in the choice and management of splitters

- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

- *Paige-Tarjan-algorithm:*

  simultaneous refinement according to 2 superblocks

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$

```
REPEAT
    B_old := B;
    FOR ALL C ∈ B_old DO B := Refine(B, C) OD
UNTIL B = B_old
return B
```

Kanellakis-Smolka algorithm

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    B_old := B;
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- maximal $|S|$ iterations

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$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

```
REPEAT
    B_old := B;
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UNTIL  B = B_old
return B
```

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs

$$\mathcal{O}(|Pre(s')| + 1)$$

Kanellakis-Smolka algorithm

```
B := B_AP; B_old := {S}        ⟵   cost: O(|S|·|AP|)
REPEAT
    B_old := B;
    FOR ALL C ∈ B_old DO B := Refine(B, C) OD
UNTIL B = B_old
return B
```

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|)$

Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

REPEAT

  $\mathcal{B}_{\mathsf{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\mathsf{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\mathsf{old}}$

return $\mathcal{B}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs

$$\mathcal{O}(|Pre(s')| + 1)$$

- cost per iteration: $\mathcal{O}(m + |S|)$

  if m = number of edges $= \sum_{s'} |Pre(s')|$

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- per iteration: each state $s' \in C$ causes the costs
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- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$
  if m = number of edges $= \sum_{s'} |Pre(s')| \geq |S|$

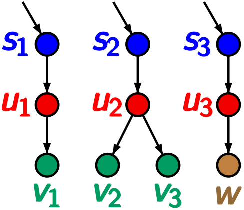
Kanellakis-Smolka algorithm

```
𝓑 := 𝓑_AP;  𝓑_old := {S}
REPEAT
    𝓑_old := 𝓑;
    FOR ALL C ∈ 𝓑_old DO 𝓑 := Refine(𝓑, C) OD
UNTIL 𝓑 = 𝓑_old
return 𝓑
```

time complexity:
$\mathcal{O}(|S| \cdot m + |S| \cdot |AP|)$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$
  if m = number of edges $= \sum_{s'} |Pre(s')| \geq |S|$

Example: Kanellakis-Smolka algorithm



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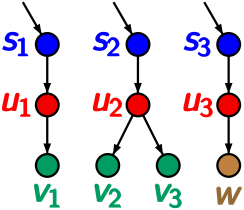
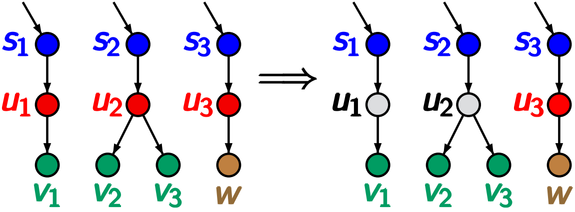


1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$

Example: Kanellakis-Smolka algorithm



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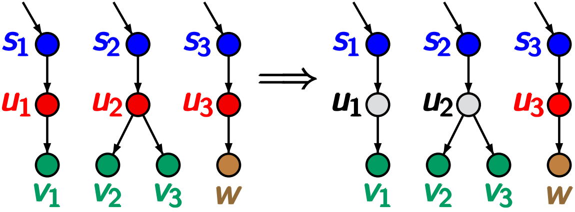
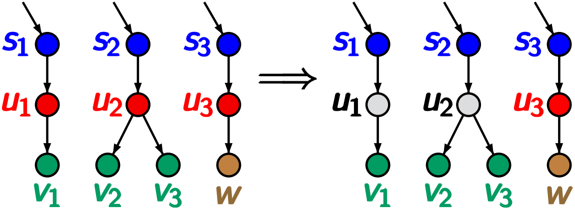


1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes

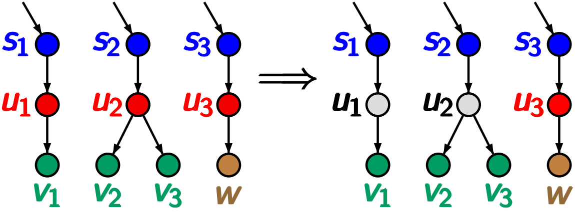
Example: Kanellakis-Smolka algorithm



1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes

Example: Kanellakis-Smolka algorithm



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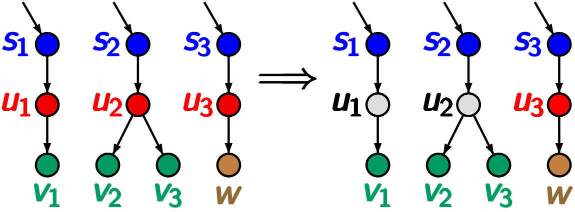
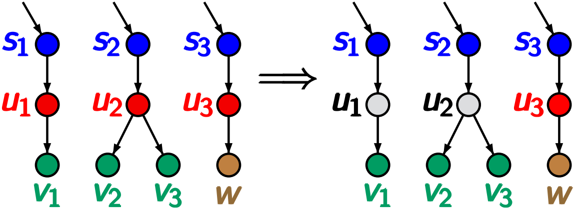
4. refinement w.r.t. $\{u_1, u_2, u_3\}$: no changes

Example: Kanellakis-Smolka algorithm



2. iteration:

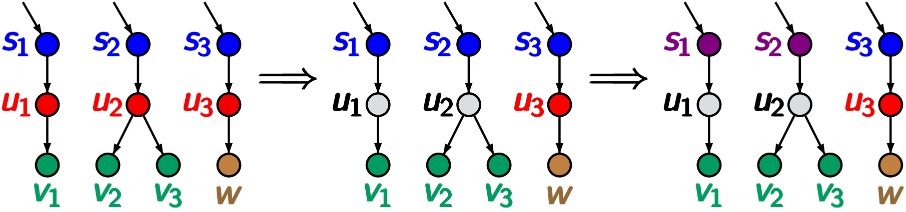
Example: Kanellakis-Smolka algorithm



2. iteration:

1. refinement w.r.t. $\{u_3\}$

Example: Kanellakis-Smolka algorithm



2. iteration:

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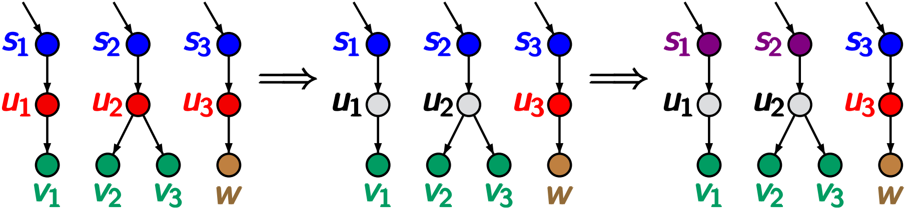
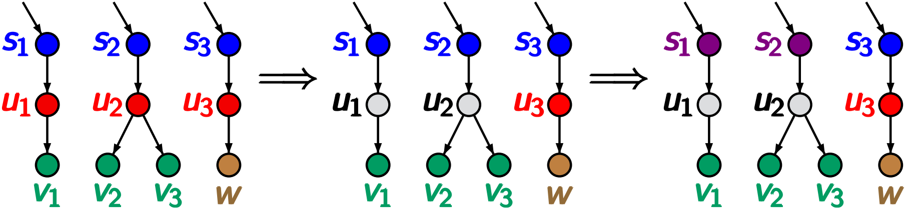
Example: Kanellakis-Smolka algorithm



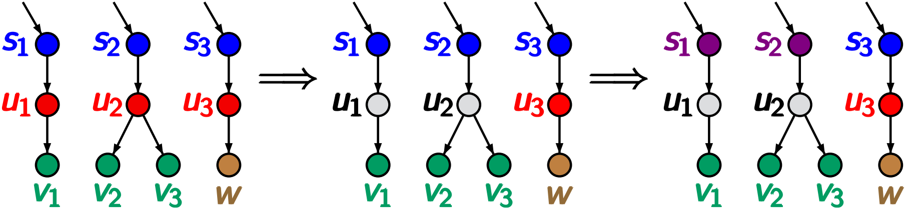
2. iteration:

1. refinement w.r.t. $\{u_3\}$
2. refinement w.r.t. other blocks of the first iteration: no changes

Example: Kanellakis-Smolka algorithm

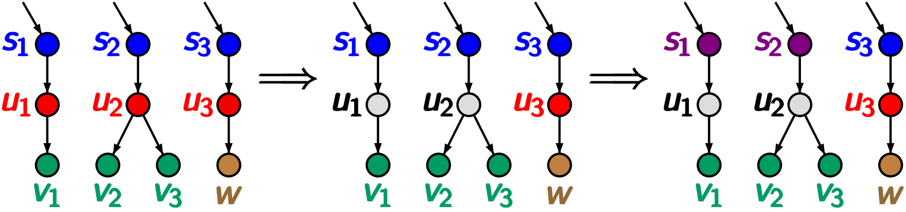


3. iteration:



3. iteration:

refinement w.r.t. all blocks of the second iteration: no changes



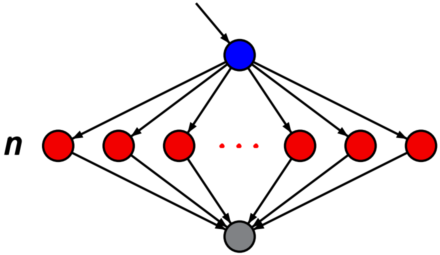
3. iteration:

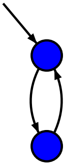
refinement w.r.t. all blocks of the second iteration: no changes

6 bisimulation equivalence classes:

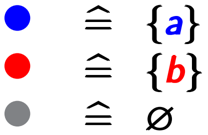
$\{s_1, s_2\}, \{s_3\}, \{u_1, u_2\}, \{u_3\}, \{v_1, v_2, v_3\}, \{w\}$

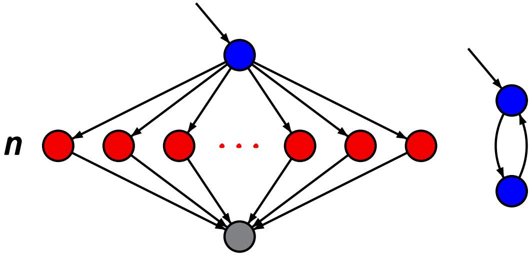
Partitioning splitter algorithm





$$AP = \{a, b\}$$





$AP = \{a, b\}$

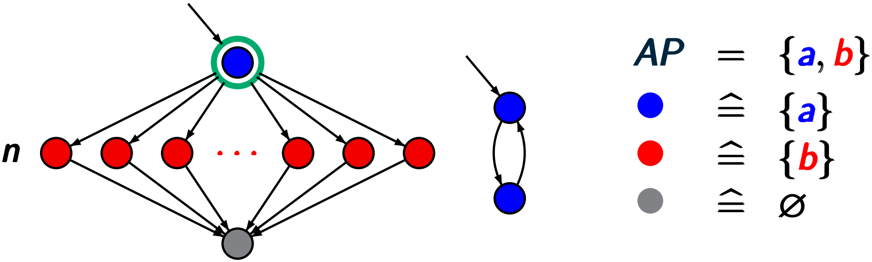
● $\widehat{=}$ $\{a\}$

● $\widehat{=}$ $\{b\}$

● $\widehat{=}$ $\varnothing$

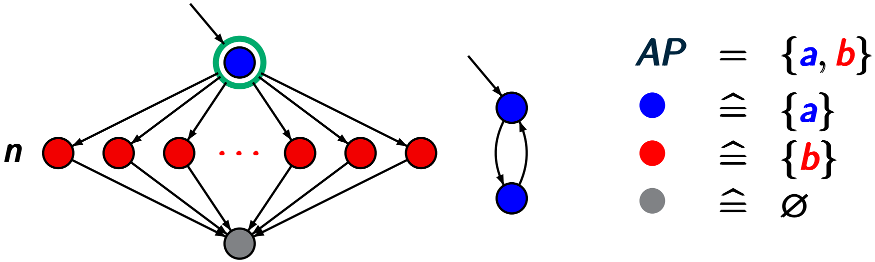
refinement w.r.t. ●:

Partitioning splitter algorithm



refinement w.r.t. :

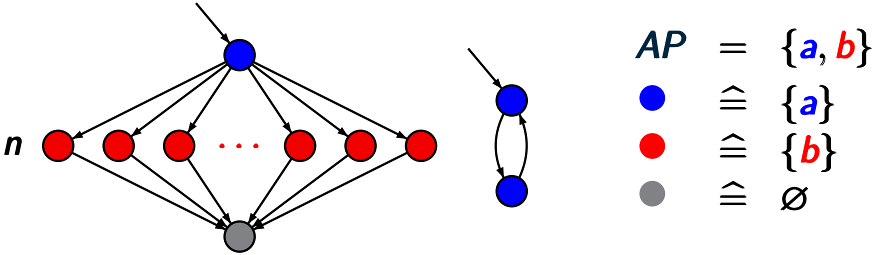
Partitioning splitter algorithm



refinement w.r.t. ●: causes the costs

$$\sum_{s'} \left|Pre(s')\right| = n$$

Partitioning splitter algorithm



refinement w.r.t. : causes the costs

$$\sum_{s'} \left| Pre(s') \right| = n$$

alternatively: refinement w.r.t. : constant costs

Partitioning splitter algorithms

Kanellakis-Smolka algorithm:

initially: $\mathcal{B}_{\mathsf{old}} = \mathcal{B} = \mathcal{B}_{AP}$

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loop invariant: $\mathcal{B}$ finer than $\mathcal{B}_{\mathsf{old}}$ and coarser than $S/\!\sim$

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Paige-Tarjan algorithm:

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iteration: ternary refinement operator

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iteration: ternary refinement operator

initially: $\mathcal{B}_{\mathsf{old}} = \{S\}$

Partitioning splitter algorithms

Kanellakis-Smolka algorithm:

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(1) $\mathcal{B}$ finer than $\mathcal{B}_{\text{old}}$ and coarser than $S/\sim$

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iteration: ternary refinement operator

initially: $\mathcal{B}_{\text{old}} = \{S\}$, $\mathcal{B} = \textit{Refine}(\mathcal{B}_{AP}, S)$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S

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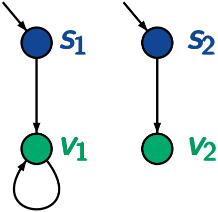

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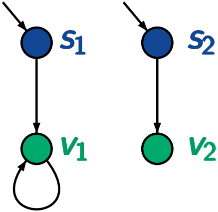
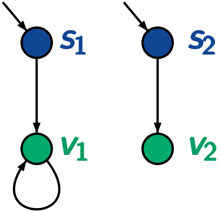


state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



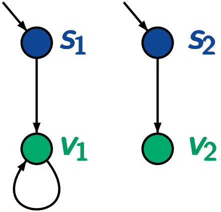
state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \big\{ \{s_1, s_2\}, \{v_1, v_2\} \big\}$$

$Pre(S)$ = set of nonterminal states

$= \{s_1, s_2, v_1\}$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



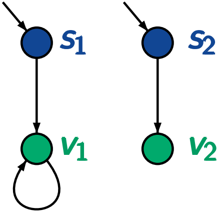
state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$Pre(S)$ = set of nonterminal states $= \{s_1, s_2, v_1\}$

$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$
$$\{v_1, v_2\} \setminus Pre(S) = \{v_2\}$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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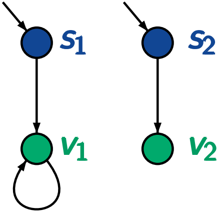
initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S)$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

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initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S) = \Big\{ \{s_1, s_2\}, \{v_1\}, \{v_2\} \Big\}$$

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

OD

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

```
WHILE B ≠ B_old DO
   select a block C' ∈ B_old \ B;



OD
```

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

```
WHILE 𝓑 ≠ 𝓑_old DO
    select a block C' ∈ 𝓑_old \ 𝓑;
    select a block C ∈ 𝓑 with C ⊆ C'
OD
```

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \textit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

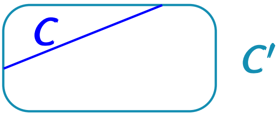
select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;



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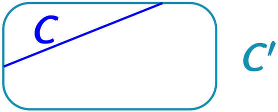
select a block $C' \in \mathcal{B}_{\mathsf{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;



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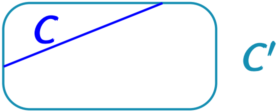

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OD

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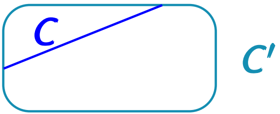

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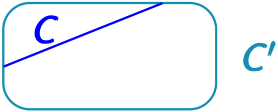

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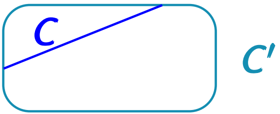

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`OD`

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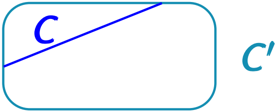

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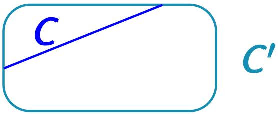

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OD

loop invariant: $\mathcal{B}$ is stable w.r.t. each block in $\mathcal{B}_{\text{old}}$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'

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simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

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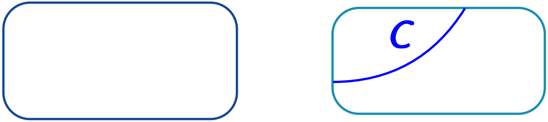

block B superblock C'

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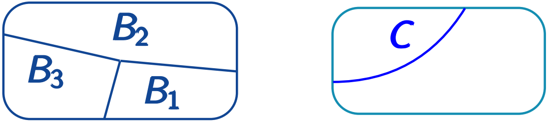

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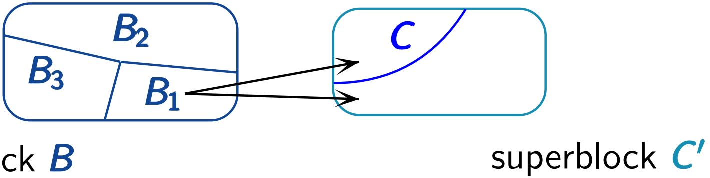

$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$

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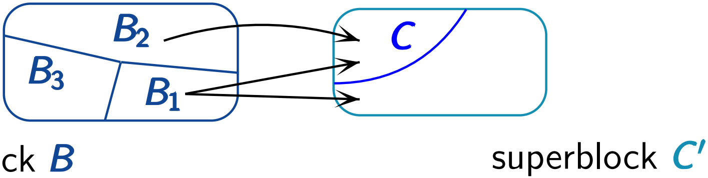

$$\begin{aligned} B_1 &= B \cap Pre(C) \cap Pre(C' \setminus C) \\ B_2 &= (B \cap Pre(C)) \setminus Pre(C' \setminus C) \end{aligned}$$

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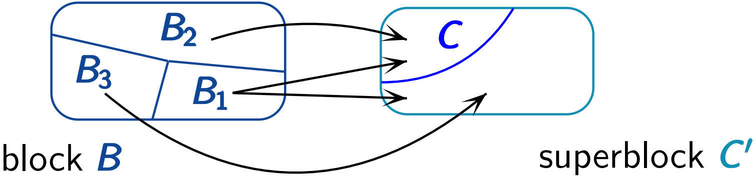

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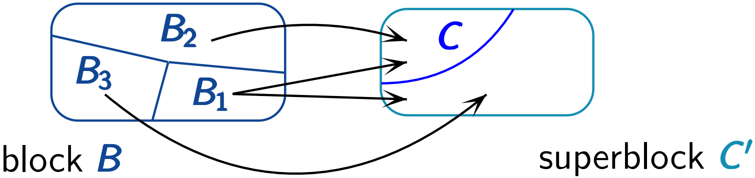

for block B with $B \cap Pre(C') = \varnothing$:

$$Refine(B, C, C' \setminus C) = \{B\}$$

Stability of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq \mathit{Pre}(C') \quad \text{or} \quad B \cap \mathit{Pre}(C') = \varnothing$$

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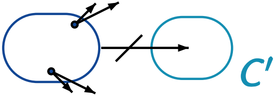
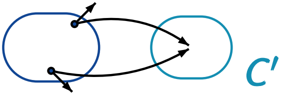


or



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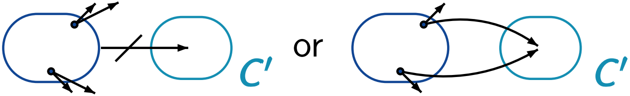
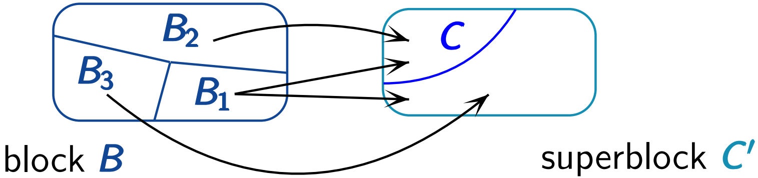


Then the new blocks B_1, B_2, B_3 in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.



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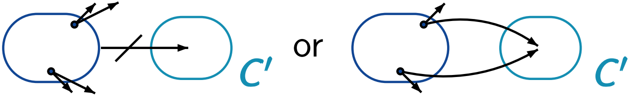
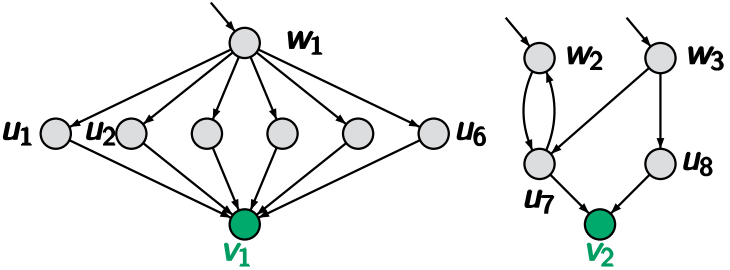


Then the new blocks B_1, B_2, B_3 in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.

If $\mathcal{B}$ is stable w.r.t. all blocks in $\mathcal{B}_{\text{old}}$ and $C' \in \mathcal{B}_{\text{old}}$, $C \in \mathcal{B}$ s.t. $C \subsetneq C'$ then $Refine(\mathcal{B}, C, C' \setminus C)$ is stable w.r.t. all blocks in the partition

$$(\mathcal{B}_{\text{old}} \setminus \{C'\}) \cup \{C, C' \setminus C\}$$

Example: Paige-Tarjan algorithm



$$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$$

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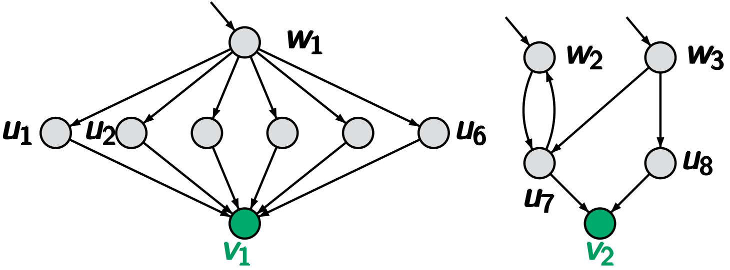

$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$

initial partition:

$$\begin{aligned} \mathcal{B}_0 &= Refine(\mathcal{B}_{AP}, S) = \mathcal{B}_{AP} \\ &= \{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\} \end{aligned}$$

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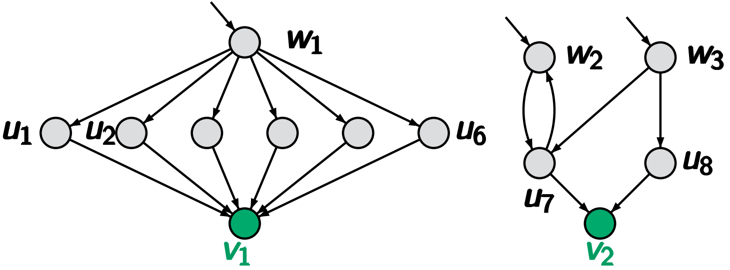

initially: $\mathcal{B}_{\text{old}} = \{S\}$

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first refinement step:

$\mathit{Refine}(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\})$

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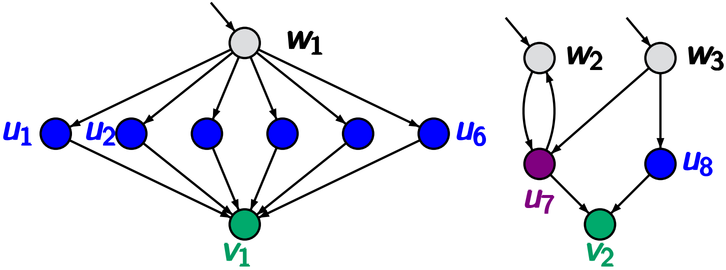

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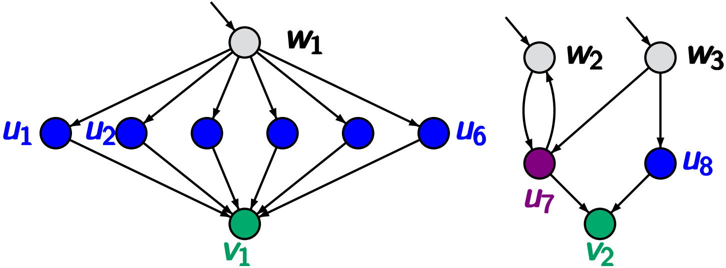

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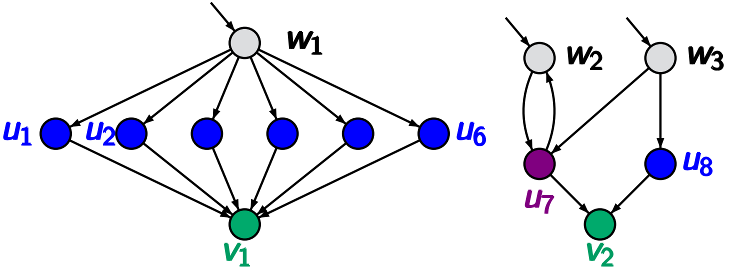


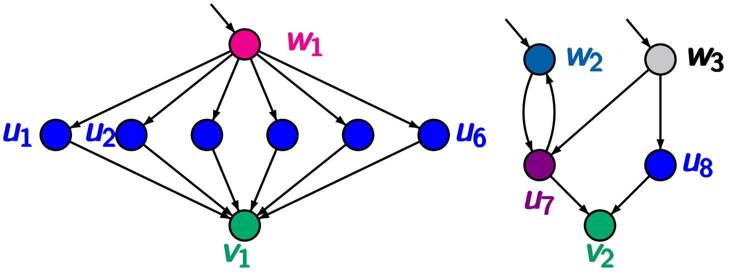
first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{\text{old}} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$Refine(\mathcal{B}_1, ?, ?)$

Example: Paige-Tarjan algorithm



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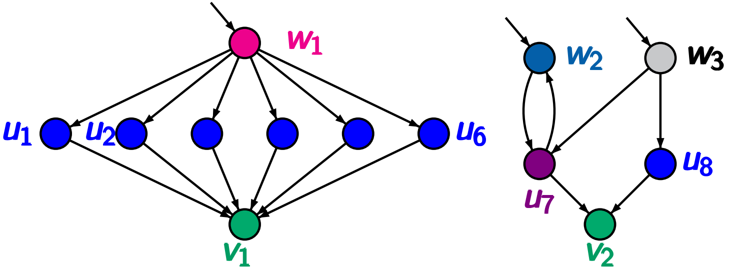

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second refinement step:

$$Refine(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$
$$= \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1\}, \{w_2\}, \{w_3\}\}$$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

return $\mathcal{B}$

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  $\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$

OD

return $\mathcal{B}$

efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;
WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO
  select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;
  add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$
  $\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$
OD
return $\mathcal{B}$

efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$ uses counters

$$\delta(s, D) = |\mathit{Post}(s) \cap D| \text{ for } D \in \mathcal{B}_{old}$$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

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$$\mathit{Refine}(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} \mathit{Refine}(B, C, C' \setminus C)$$

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$D \in \mathcal{B}_{\mathsf{old}}$

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$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$$\underbrace{s \in Pre(D)}_{\text{for } s} \qquad \underbrace{D \in \mathcal{B}_{\text{old}}}_{\text{for } D}$$

Details of the Paige-Tarjan algorithm

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$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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$$s \in Pre(D) \qquad D \in \mathcal{B}_{\mathsf{old}}$$

step 1: compute $\delta(\ldots)$ for the new blocks C and $C' \setminus C$ in $\mathcal{B}_{\mathsf{old}}$

Details of the Paige-Tarjan algorithm

implementation of

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step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C) \leftarrow \boxed{\text{for } s \in Pre(C')}$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\begin{aligned} \delta(s, C) &= |\mathit{Post}(s) \cap C| \\ \delta(s, C' \setminus C) &= |\mathit{Post}(s) \cap (C' \setminus C)| \end{aligned}$$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

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step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \cap \mathit{Pre}(C') = \varnothing$ we have:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

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for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ $\boxed{\text{for } s \in Pre(C')}$

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> for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:
>
> $$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) = |Post(s) \cap C|$$
$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

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$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$
$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$
$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C) \leftarrow \boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

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$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$

$$B_3 = \{s \in B : \delta(s, C) = 0, \delta(s, C' \setminus C) > 0\}$$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \backslash C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

`FOR ALL` $s \in S$ `DO` $\delta(s, S) := |\mathit{Post}(s)|$ `OD`

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ `DO`

  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add C and $C' \backslash C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

`OD`

Paige-Tarjan algorithm

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

FOR ALL $s \in S$ DO $\delta(s, S) := |Post(s)|$ OD

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

  select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add C and $C' \backslash C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

    FOR ALL $s \in Pre(C)$ DO $\delta(s, C) := 0$ OD

    FOR ALL $s' \in C$ DO

      FOR ALL $s \in Pre(s')$ DO $\delta(s, C) := \delta(s, C) + 1$ OD

    OD

  $\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

OD

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;
FOR ALL $s \in S$ DO $\delta(s, S) := |\mathit{Post}(s)|$ OD
WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO
  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;
  add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$
    FOR ALL $s \in \mathit{Pre}(C)$ DO $\delta(s, C) := \mathbf{0}$ OD
    FOR ALL $s' \in C$ DO
      FOR ALL $s \in \mathit{Pre}(s')$ DO $\delta(s, C) := \delta(s, C) + \mathbf{1}$ OD
    OD
    FOR ALL $s \in \mathit{Pre}(C)$ DO
        $\delta(s, C' \setminus C) := \delta(s, C') - \delta(s, C)$ OD
  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$
OD

Complexity of the Paige-Tarjan algorithm

Complexity of the Paige-Tarjan algorithm

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

$$n = \#\text{ states} \qquad = |S|$$

$$m = \#\text{ transitions}$$

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Complexity of the Paige-Tarjan algorithm

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

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in what follows, we suppose $m \geq n$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$;

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WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and
remove C' from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S);$ $\longleftarrow$ $\boxed{\text{complexity: } \mathcal{O}(n \cdot |AP|)}$

$\mathcal{B}_{\mathsf{old}} := \{S\};$

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and
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OD

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```
WHILE
```
$\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$
```
DO
```

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

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$$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$$

```
OD
```

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\longleftarrow$ $\boxed{\text{complexity: } \mathcal{O}(n \cdot |AP|)}$

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add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and
remove C' from $\mathcal{B}_{\mathsf{old}}$

$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\sum_{s' \in C} |\mathit{Pre}(s')| + 1$

OD

Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

$\mathcal{B}_{old} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t.
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add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and
remove C' from $\mathcal{B}_{old}$

$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\mathcal{O}(|C| + |\mathit{Pre}(C)|)$

OD

Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

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add C and $C' \backslash C$ to $\mathcal{B}_{old}$ and
remove C' from $\mathcal{B}_{old}$

$\boxed{\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\mathcal{O}(|C| + |Pre(C)|)$

OD

total cost for all refinement operations: $\mathcal{O}(m \cdot \log n)$