

- **linear** vs. **branching time**
 - * linear time: trace relations
 - * branching time: (bi)simulation relations
- **(nonsymmetric) preorders** vs. **equivalences**:
 - * preorders: trace inclusion, simulation
 - * equivalences: trace equivalence, bisimulation
- **strong** vs. **weak** relations
 - * strong: reasoning about **all transitions**
 - * weak: abstraction from **stutter steps**

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

bisimulation



CTL, CTL*-equivalence

computing the bisimulation quotient

abstraction stutter steps

simulation relations

let $\mathcal{T}_1 = (S_1, Act_1, \rightarrow_1, S_{0,1}, AP, L_1)$,
 $\mathcal{T}_2 = (S_2, Act_2, \rightarrow_2, S_{0,2}, AP, L_2)$

be two transition systems

- with the same set AP
- possibly containing terminal states

Bisimulation equivalence of \mathcal{T}_1 and \mathcal{T}_2 requires that \mathcal{T}_1 and \mathcal{T}_2 can simulate each other in a stepwise manner.

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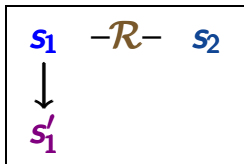
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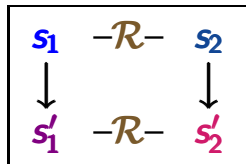
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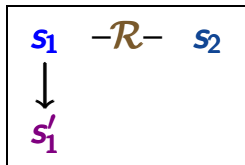
can be
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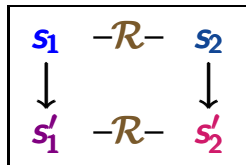
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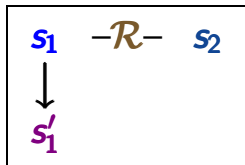


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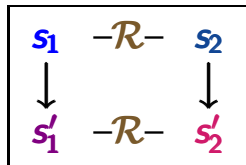
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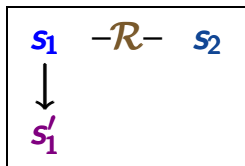
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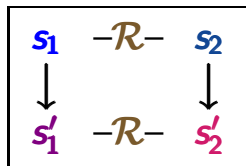
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bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$: relation $\mathcal{R} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ s.t.

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- (2) } mutual stepwise
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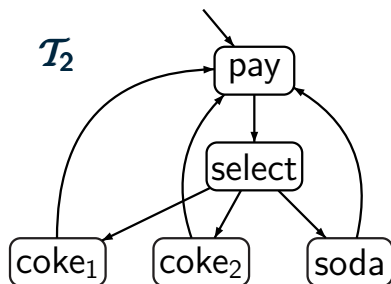
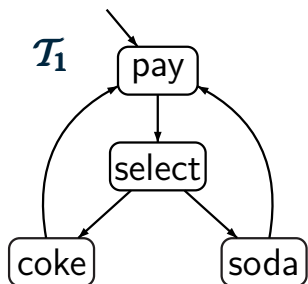
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for state s_1 of \mathcal{T}_1 and state s_2 of \mathcal{T}_2 :

$s_1 \sim s_2$ iff there exists a bisimulation \mathcal{R} for $(\mathcal{T}_1, \mathcal{T}_2)$
such that $(s_1, s_2) \in \mathcal{R}$

Two beverage machines

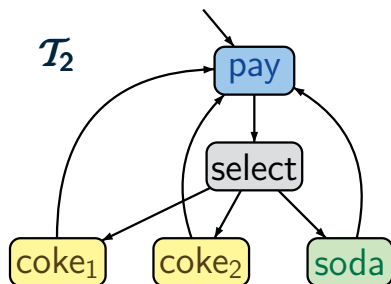
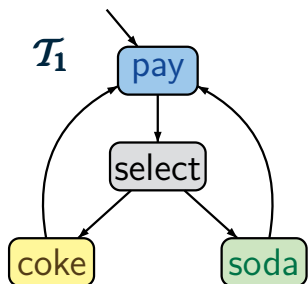
BSEQOR5.1-8-BIS



$$AP = \{pay, coke, soda\}$$

Two beverage machines

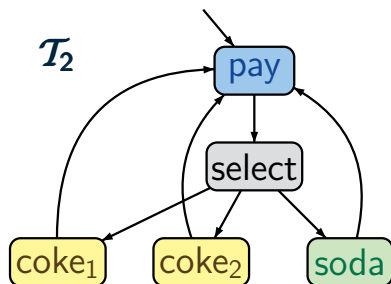
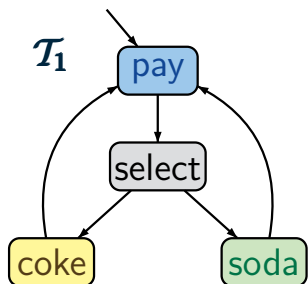
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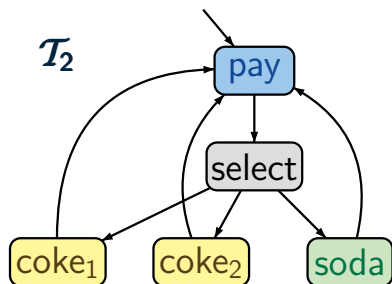
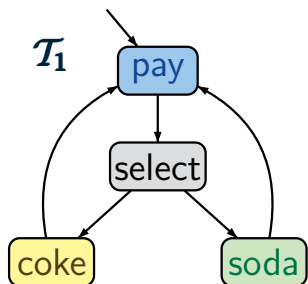


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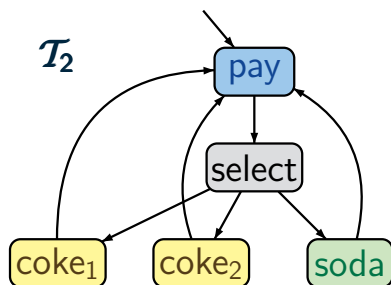
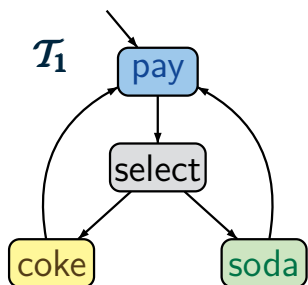


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Two beverage machines

BSEQOR5.1-8-BIS



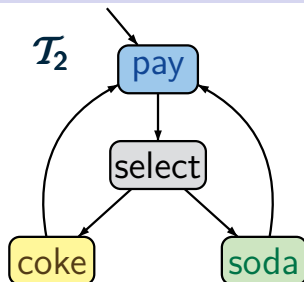
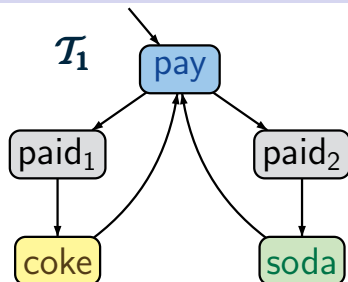
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$$\left\{ \begin{array}{l} (\text{pay}, \text{pay}), (\text{select}, \text{select}), (\text{soda}, \text{soda}) \\ (\text{coke}, \text{coke}_1), (\text{coke}, \text{coke}_2) \end{array} \right\}$$

Two beverage machines

BSEQOR5.1-8-BIS-3

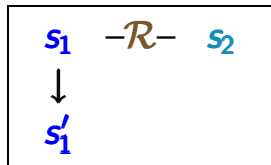


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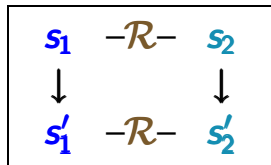
$\mathcal{T}_1 \not\sim \mathcal{T}_2$

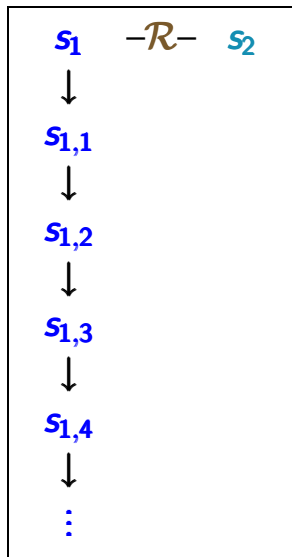
because there is no state in \mathcal{T}_1 that has both

- a successor labeled with **coke** and
- a successor labeled with **soda**



can be
completed to

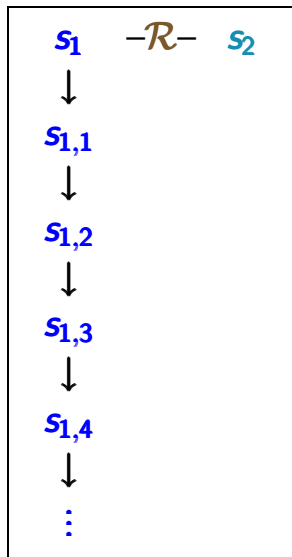




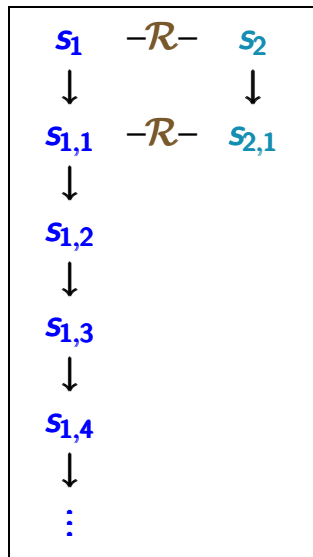
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Path lifting for bisimulation \mathcal{R}

BSEQOR5.1-9-BIS

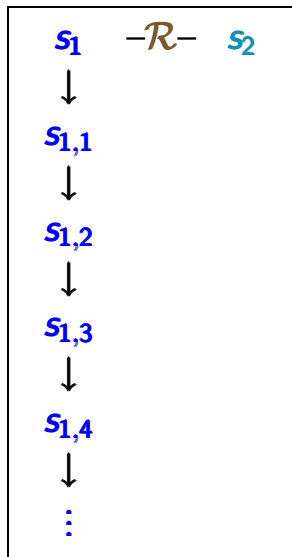


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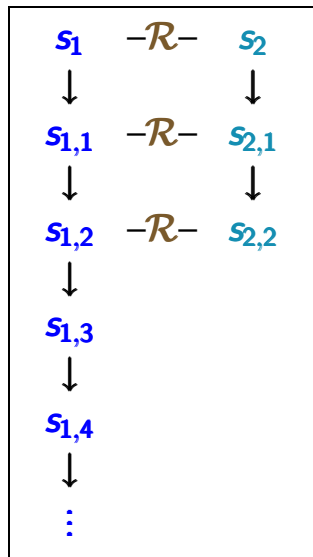


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BSEQOR5.1-9-BIS

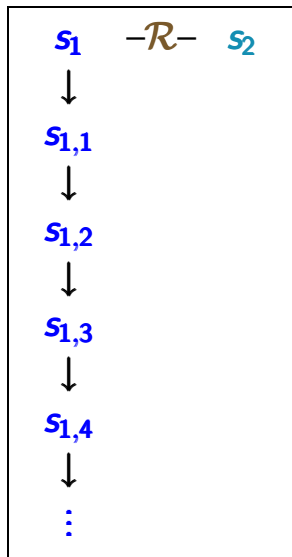


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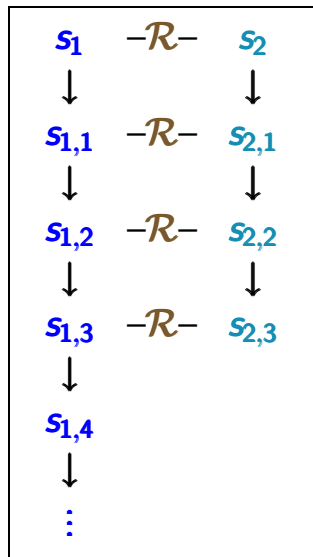


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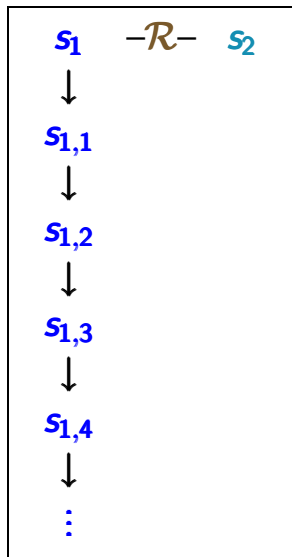


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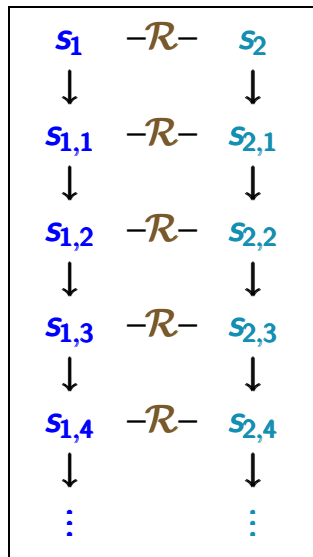


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$$\mathcal{R} = \{(s, s) : s \in S\}$$

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$$\mathcal{R}^{-1} = \{(s_2, s_1) : (s_1, s_2) \in \mathcal{R}\}$$

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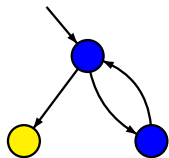
$\mathcal{R}_{2,3}$ be a bisimulation for $(\mathcal{T}_2, \mathcal{T}_3)$.

$$\mathcal{R} \stackrel{\text{def}}{=} \left\{ (s_1, s_3) : \exists s_2 \text{ s.t. } (s_1, s_2) \in \mathcal{R}_{1,2} \right. \\ \left. \text{and } (s_2, s_3) \in \mathcal{R}_{2,3} \right\}$$

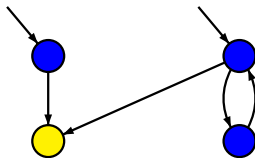
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Correct or wrong?

BSEQOR5.1-19

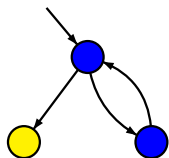


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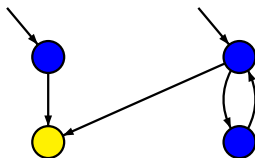


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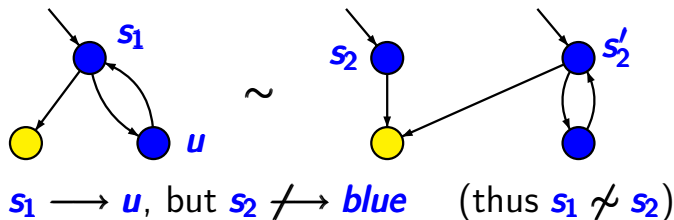
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wrong

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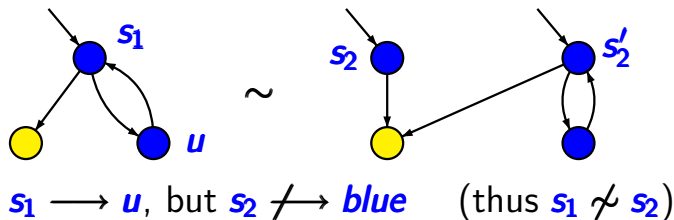
BSEQOR5.1-19



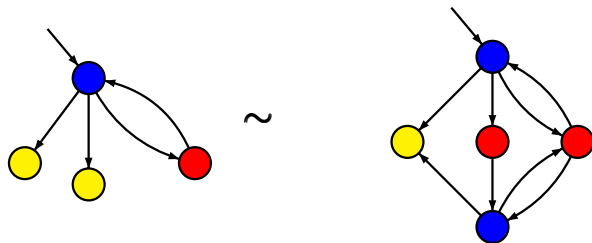
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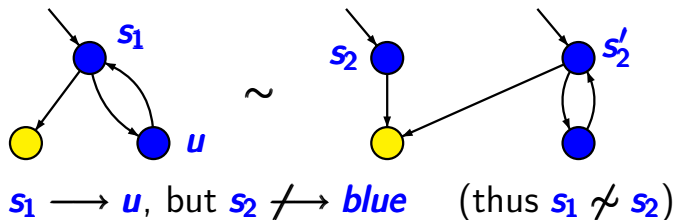


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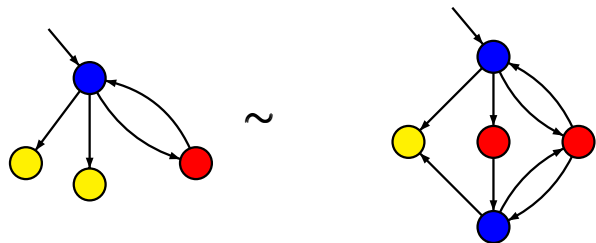


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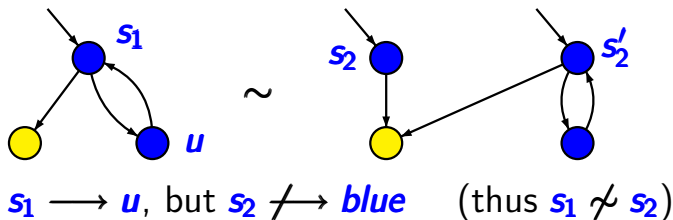
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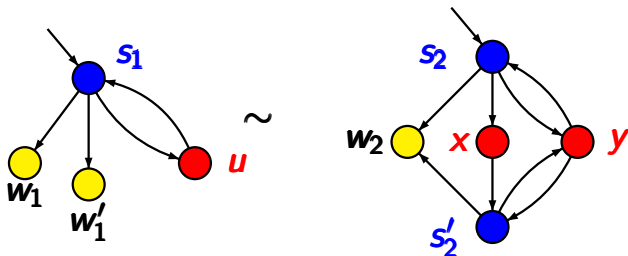
correct

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BSEQOR5.1-19



wrong

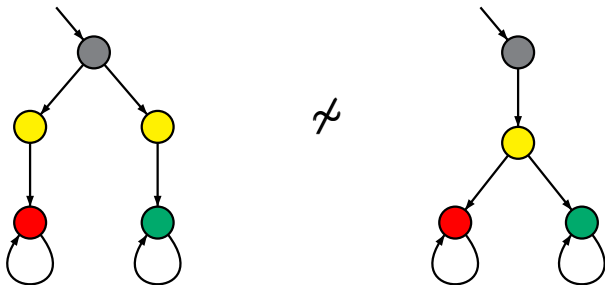


correct

$$\mathcal{T}_1 \sim \mathcal{T}_2 \implies \text{Traces}(\mathcal{T}_1) = \text{Traces}(\mathcal{T}_2)$$

proof: ... path fragment lifting ...

$$\text{Traces}(\mathcal{T}_1) = \text{Traces}(\mathcal{T}_2) \not\implies \mathcal{T}_1 \sim \mathcal{T}_2$$



trace equivalent, but not bisimulation equivalent

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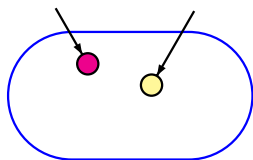
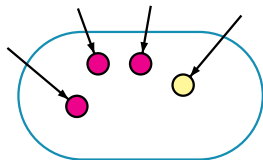
$$\text{Traces}(\mathcal{T}_1) = \text{Traces}(\mathcal{T}_2) \not\Rightarrow \mathcal{T}_1 \sim \mathcal{T}_2$$

Trace equivalence is **strictly coarser** than
bisimulation equivalence.

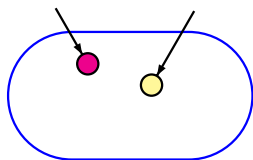
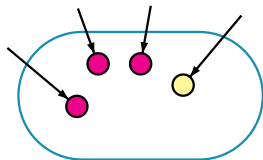
Bisimulation equivalent transition systems satisfy
the **same LT properties** (e.g., **LTL formulas**).

- as a relation that compares **2** transition systems

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 \mathcal{T}_1  \mathcal{T}_2 

- as a relation that compares **2** transition systems

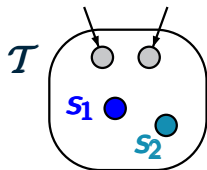
 \mathcal{T}_1  \mathcal{T}_2 

- as a relation on the **states** of **1** transition system

- as a relation that compares **2** transition systems



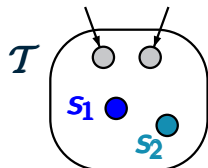
- as a relation on the **states** of **1** transition system



- as a relation that compares **2** transition systems



- as a relation on the **states** of **1** transition system

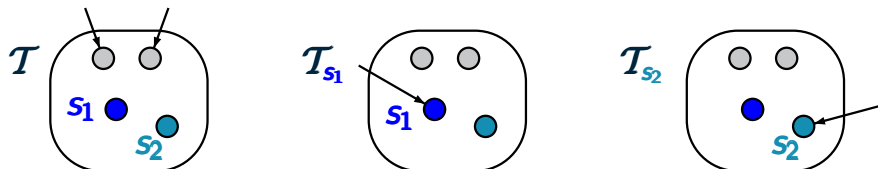


$$s_1 \sim s_2 \text{ iff } \mathcal{T}_{s_1} \sim \mathcal{T}_{s_2}$$

- as a relation that compares **2** transition systems



- as a relation on the **states** of **1** transition system

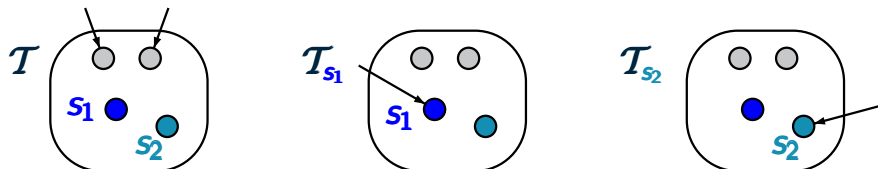


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- as a relation that compares **2** transition systems



- as a relation on the **states** of **1** transition system



$s_1 \sim s_2$ iff $\mathcal{T}_{s_1} \sim \mathcal{T}_{s_2}$ iff
 there exists a bisimulation \mathcal{R} for \mathcal{T} s.t. $(s_1, s_2) \in \mathcal{R}$

Let \mathcal{T} be a TS with proposition set AP .

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A **bisimulation** for \mathcal{T} is a binary relation \mathcal{R} on the state space of \mathcal{T} s.t. for all $(s_1, s_2) \in \mathcal{R}$:

- (1) $L(s_1) = L(s_2)$
- (2) $\forall s'_1 \in \text{Post}(s_1) \exists s'_2 \in \text{Post}(s_2)$ s.t. $(s'_1, s'_2) \in \mathcal{R}$
- (3) $\forall s'_2 \in \text{Post}(s_2) \exists s'_1 \in \text{Post}(s_1)$ s.t. $(s'_1, s'_2) \in \mathcal{R}$

Let \mathcal{T} be a TS with proposition set AP .

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bisimulation equivalence $\sim_{\mathcal{T}}$:

$s_1 \sim_{\mathcal{T}} s_2$ iff there exists a bisimulation \mathcal{R} for \mathcal{T}
s.t. $(s_1, s_2) \in \mathcal{R}$

Let \mathcal{T} be a TS with proposition set AP .

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coinductive definition of $\sim_{\mathcal{T}}$:

$s_1 \sim_{\mathcal{T}} s_2$ iff there exists a bisimulation \mathcal{R} for \mathcal{T}
s.t. $(s_1, s_2) \in \mathcal{R}$

Let \mathcal{T} be a transition system with state space \mathcal{S} .

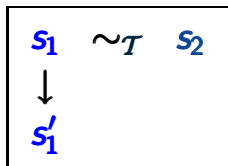
Bisimulation equivalence $\sim_{\mathcal{T}}$ is

- the coarsest bisimulation on \mathcal{T}
- and an equivalence on \mathcal{S}

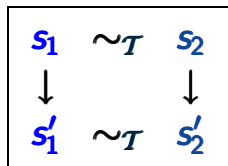
Let \mathcal{T} be a transition system with state space \mathcal{S} .

Bisimulation equivalence $\sim_{\mathcal{T}}$ is the **coarsest equivalence** on \mathcal{S} s.t. for all states $s_1, s_2 \in \mathcal{S}$ with $s_1 \sim_{\mathcal{T}} s_2$:

- (1) $L(s_1) = L(s_2)$
- (2) each transition of s_1 can be mimicked by a transition of s_2 :



can be
completed to




Two variants of bisimulation equivalence

- \sim relation that compares **2** transition systems
- $\sim_{\mathcal{T}}$ equivalence on the state space of a single TS \mathcal{T}

1. $\sim_{\mathcal{T}}$ can be derived from \sim

for all states s_1 and s_2 of \mathcal{T} :

$$s_1 \sim_{\mathcal{T}} s_2 \text{ iff } \mathcal{T}_{s_1} \sim \mathcal{T}_{s_2}$$



where \mathcal{T}_s agrees with \mathcal{T} , except that state s is declared to be the unique initial state

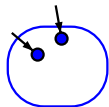
2. \sim can be derived from $\sim_{\mathcal{T}}$

Derivation of \sim from $\sim_{\mathcal{T}}$

BSEQOR5.1-31

given two transition systems \mathcal{T}_1 and \mathcal{T}_2

\mathcal{T}_1 with state space S_1

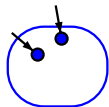


\mathcal{T}_2 with state space S_2



given two transition systems \mathcal{T}_1 and \mathcal{T}_2

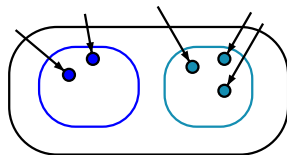
\mathcal{T}_1 with state space S_1



\mathcal{T}_2 with state space S_2

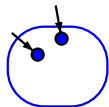


consider $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2$
(state space $S_1 \uplus S_2$)



given two transition systems \mathcal{T}_1 and \mathcal{T}_2

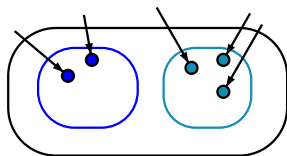
\mathcal{T}_1 with state space S_1



\mathcal{T}_2 with state space S_2



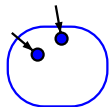
consider $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2$
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$\mathcal{T}_1 \sim \mathcal{T}_2$ iff \forall initial states s_1 of \mathcal{T}_1
 \exists initial state s_2 of \mathcal{T}_2 s.t. $s_1 \sim_{\mathcal{T}} s_2$,

given two transition systems \mathcal{T}_1 and \mathcal{T}_2

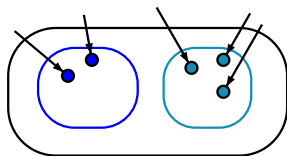
\mathcal{T}_1 with state space S_1



\mathcal{T}_2 with state space S_2



consider $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2$
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$\mathcal{T}_1 \sim \mathcal{T}_2$ iff \forall initial states s_1 of \mathcal{T}_1
 \exists initial state s_2 of \mathcal{T}_2 s.t. $s_1 \sim_{\mathcal{T}} s_2$,
 and vice versa

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a TS.

Let $\mathcal{T} = (\mathcal{S}, Act, \rightarrow, S_0, AP, L)$ be a TS.

bisimulation quotient \mathcal{T}/\sim arises from \mathcal{T}
by collapsing bisimulation equivalent states

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a TS.

bisimulation quotient:

$$\mathcal{T}/\sim = (S', Act', \rightarrow', S'_0, AP, L')$$

Let $\mathcal{T} = (\mathcal{S}, \text{Act}, \rightarrow, \mathcal{S}_0, AP, L)$ be a TS.

bisimulation quotient:

$$\mathcal{T}/\sim = (\mathcal{S}', \text{Act}', \rightarrow', \mathcal{S}'_0, AP, L')$$

- state space: $\mathcal{S}' = \mathcal{S}/\sim_{\mathcal{T}}$



set of bisimulation equivalence classes

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a TS.

bisimulation quotient:

$$\mathcal{T}/\sim = (S', Act', \rightarrow', S'_0, AP, L')$$

- state space: $S' = S/\sim_{\mathcal{T}}$
- set of initial states: $S'_0 = \{[s]_{\sim_{\mathcal{T}}} : s \in S_0\}$

Let $\mathcal{T} = (\mathcal{S}, \text{Act}, \rightarrow, \mathcal{S}_0, AP, L)$ be a TS.

bisimulation quotient:

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- labeling function: $L'([s]_{\sim_{\mathcal{T}}}) = L(s)$

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well-defined

by the labeling condition
of bisimulations

Let $\mathcal{T} = (\mathcal{S}, \text{Act}, \rightarrow, \mathcal{S}_0, \text{AP}, L)$ be a TS.

bisimulation quotient:

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- labeling function: $L'([s]_{\sim_{\mathcal{T}}}) = L(s)$
- transition relation:

$$\frac{s \longrightarrow s'}{[s]_{\sim_{\mathcal{T}}} \longrightarrow [s']_{\sim_{\mathcal{T}}}}$$

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action labels
irrelevant

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$$\frac{s \xrightarrow{\alpha} s'}{[s]_{\sim_{\mathcal{T}}} \xrightarrow{\mathcal{T}} [s']_{\sim_{\mathcal{T}}}}$$

$$\mathcal{T} \sim \mathcal{T}/\sim$$

Example: interleaving of n printers

BSEQOR5.1-34

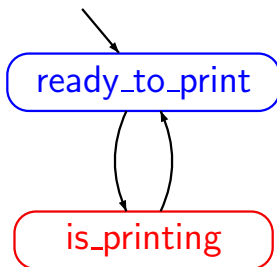
parallel system $\mathcal{T} = \underbrace{Printer \parallel Printer \parallel \dots \parallel Printer}_{n \text{ printers}}$

Example: interleaving of n printers

BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{Printer \parallel\parallel Printer \parallel\parallel \dots \parallel\parallel Printer}_{n \text{ printers}}$

transition system
for each printer



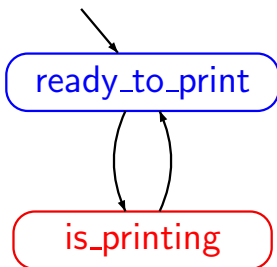
Example: interleaving of n printers

BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{Printer \parallel \dots \parallel Printer}_{n \text{ printers}}$

$AP = \{0, 1, \dots, n\}$ “number of available printers”

transition system
for each printer

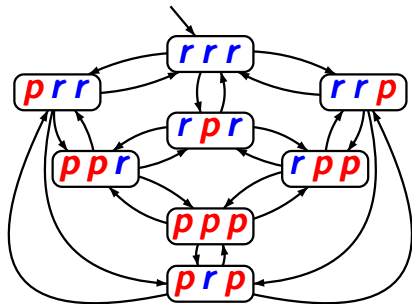


Example: $n=3$ printers

BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{Printer \parallel\parallel Printer \parallel\parallel \dots \parallel\parallel Printer}_{n \text{ printers}}$

$AP = \{0, 1, 2, 3\}$



p : is printing

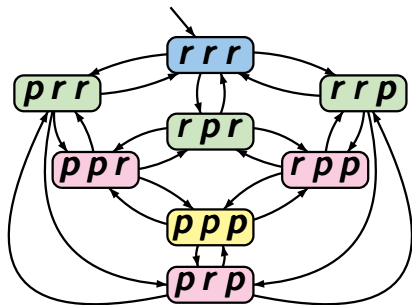
r : ready to print

Example: $n=3$ printers

BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{Printer \parallel\parallel Printer \parallel\parallel \dots \parallel\parallel Printer}_{n \text{ printers}}$

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p : is printing

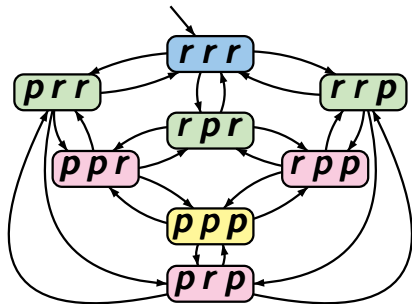
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Example: $n=3$ printers

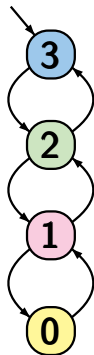
BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{\text{Printer} \parallel \text{Printer} \parallel \dots \parallel \text{Printer}}_{n \text{ printers}}$

$AP = \{0, 1, 2, 3\}$



p : is printing
 r : ready to print



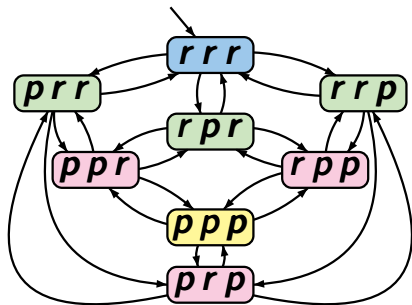
bisimulation
quotient

Example: $n=3$ printers

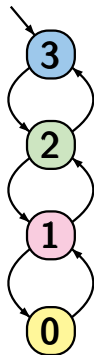
BSEQOR5.1-34

parallel system $\mathcal{T} = \underbrace{Printer \parallel\parallel Printer \parallel\parallel \dots \parallel\parallel Printer}_{n \text{ printers}}$

$AP = \{0, 1, 2, 3\}$



2^n states



$n+1$ states

solutions for mutual exclusion problems:

- semaphore
- Peterson's algorithm
- Bakery algorithm



given two concurrent processes P_1 and P_2

- two additional shared variables: $x_1, x_2 \in \mathbb{N}$
- if P_1 and P_2 are waiting then:
 - if $x_1 < x_2$ then P_1 enters its critical section
 - if $x_2 < x_1$ then P_2 enters its critical section
 - $x_1 = x_2$: cannot happen

protocol for P_1 :

```
LOOP FOREVER
```

```
  noncritical actions
```

```
   $x_1 := x_2 + 1$ 
```

```
  AWAIT ( $x_1 < x_2$ )  $\vee$  ( $x_2 = 0$ );
```

```
  critical section;
```

```
   $x_1 := 0$ 
```

```
END LOOP
```

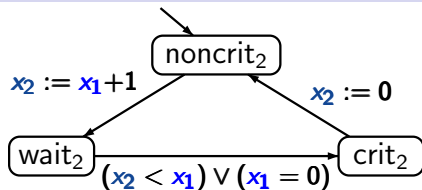
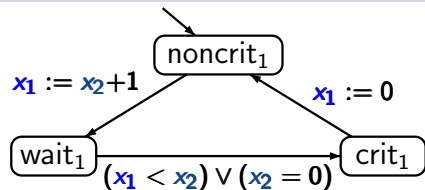
initially:

```
 $x_1 = x_2 = 0$ 
```

symmetric protocol for P_2

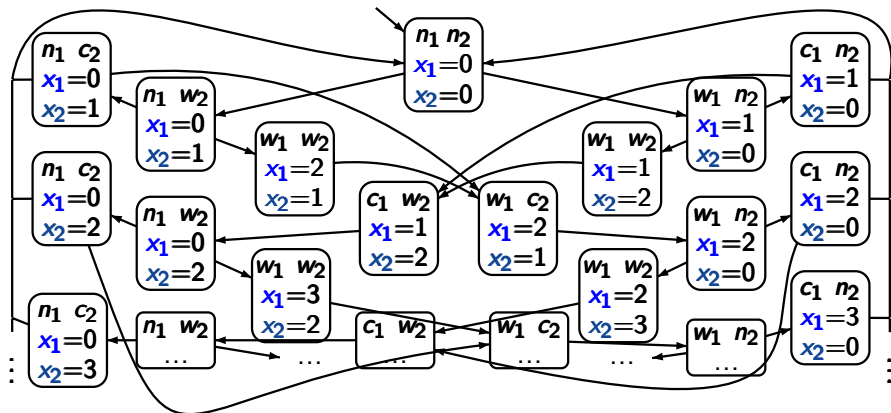
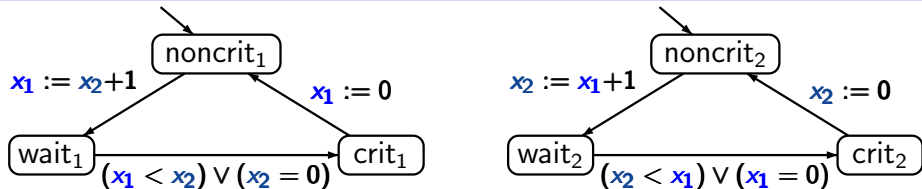
Program graphs for the Bakery algorithm

BSEQOR5.1-37



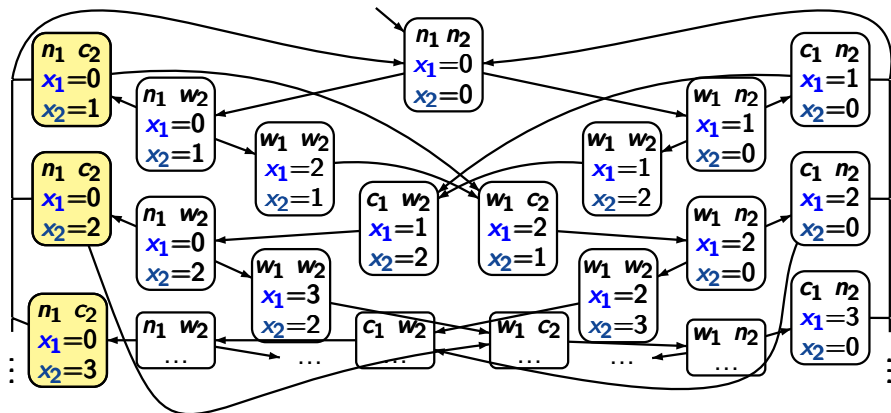
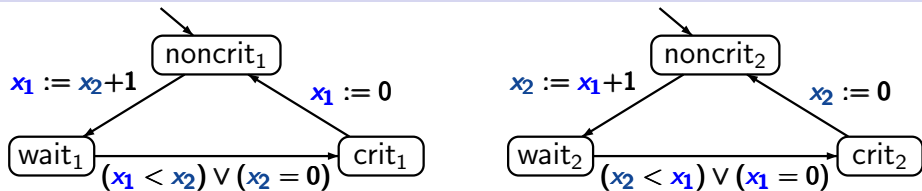
Transition system for the Bakery algorithm

BSEQOR5.1-37



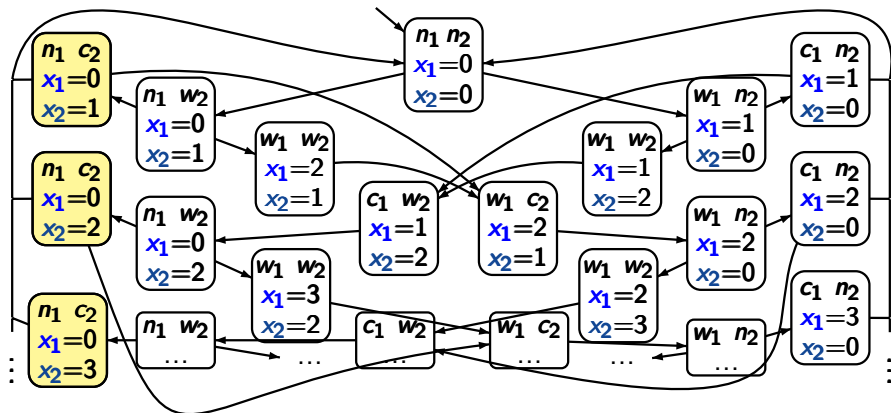
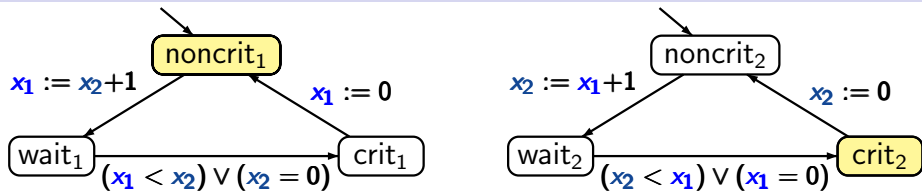
Transition system for the Bakery algorithm

BSEQOR5.1-37



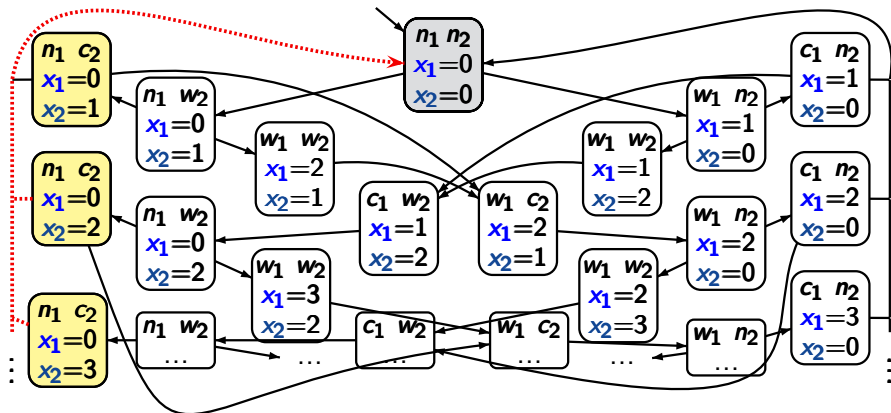
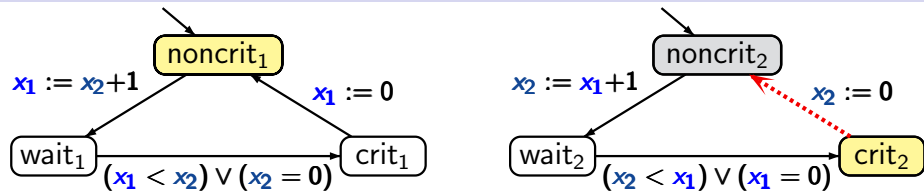
Transition system for the Bakery algorithm

BSEQOR5.1-37



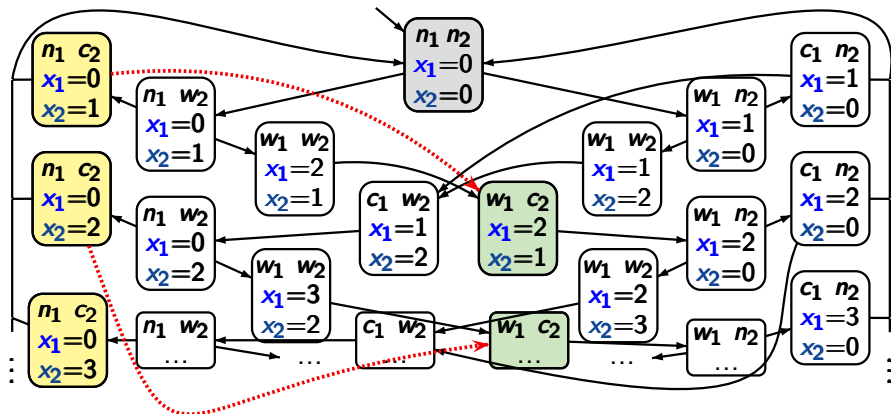
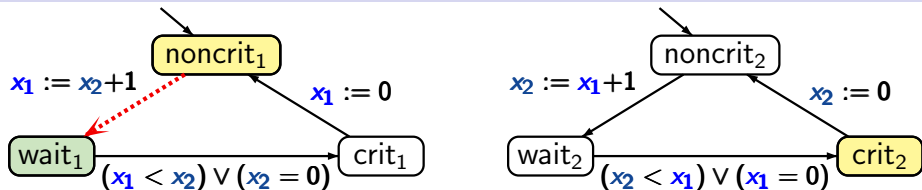
Transition system for the Bakery algorithm

BSEQOR5.1-37



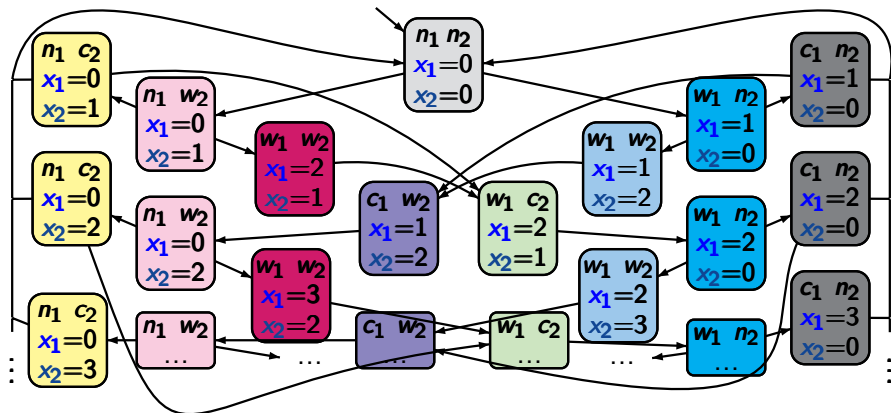
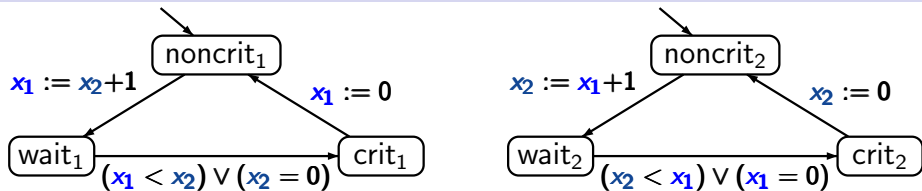
Transition system for the Bakery algorithm

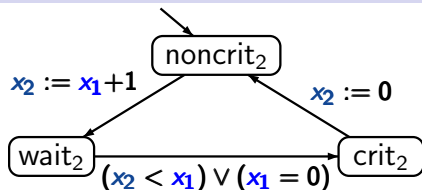
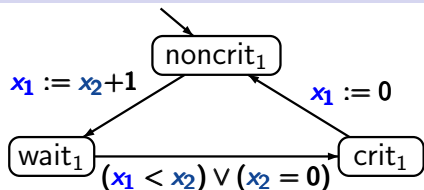
BSEQOR5.1-37



Transition system for the Bakery algorithm

BSEQOR5.1-37

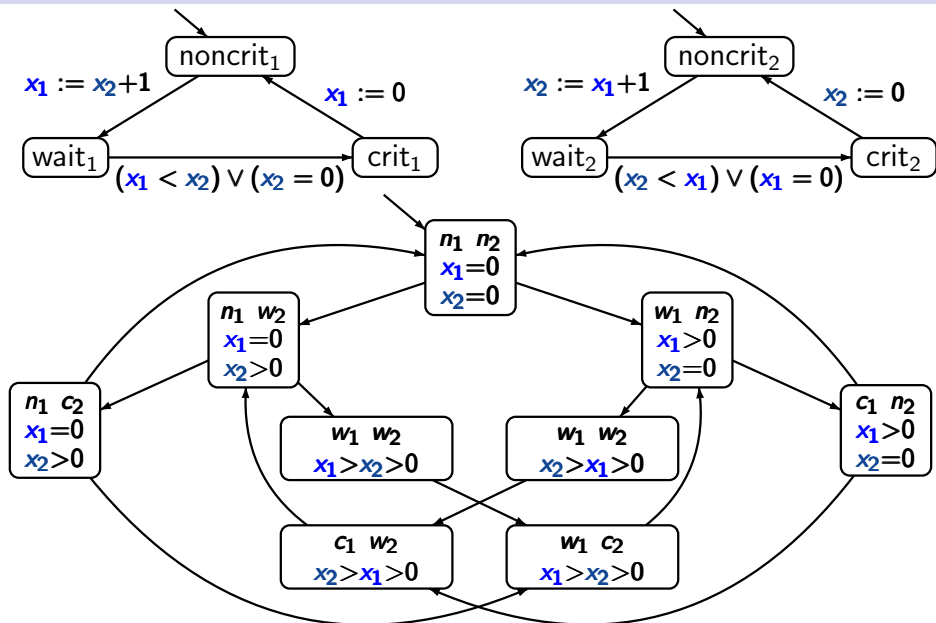




infinite transition system with a
finite bisimulation quotient

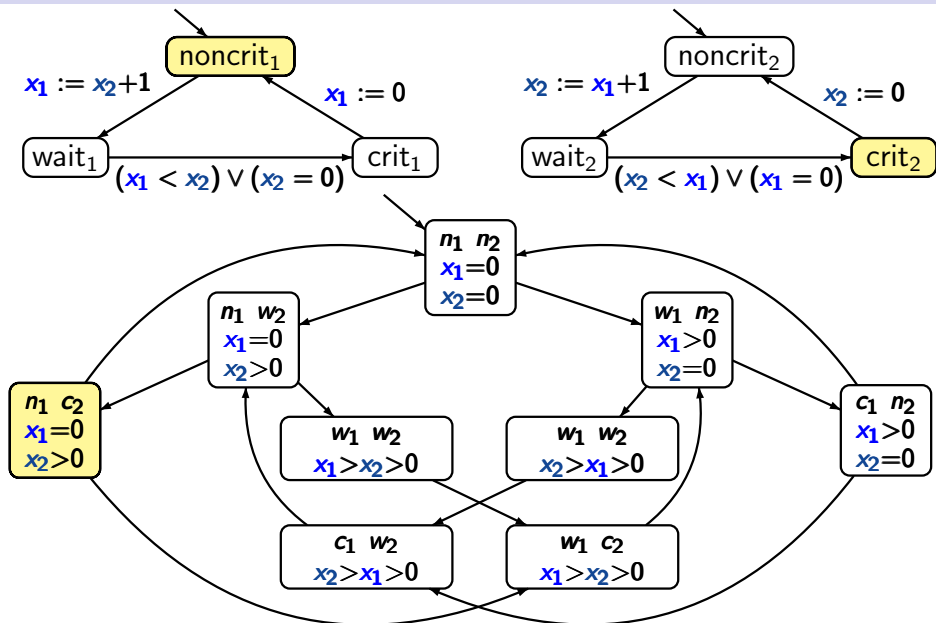
Bakery algorithm: bisimulation quotient

BSEQOR5.1-38



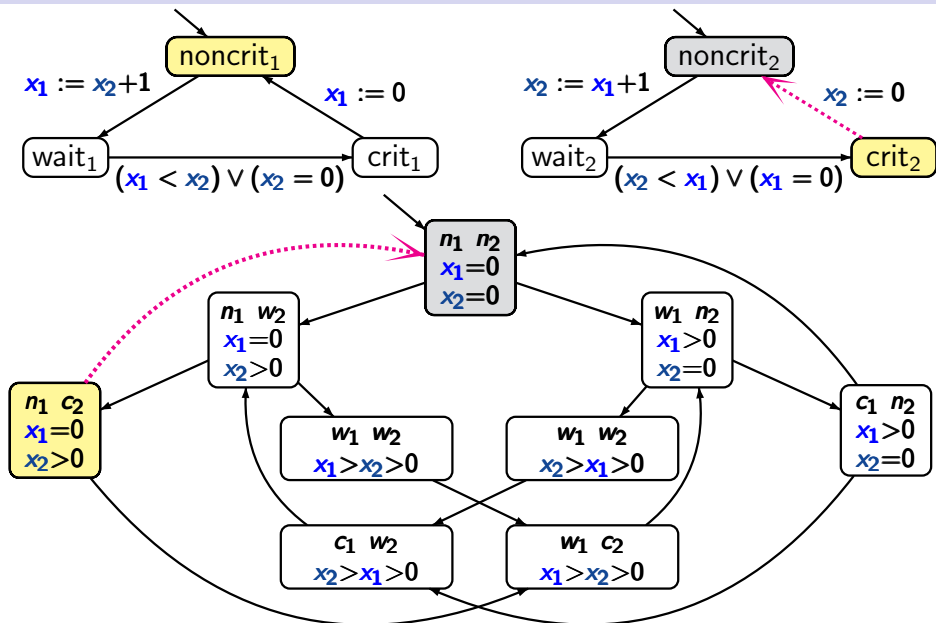
Bakery algorithm: bisimulation quotient

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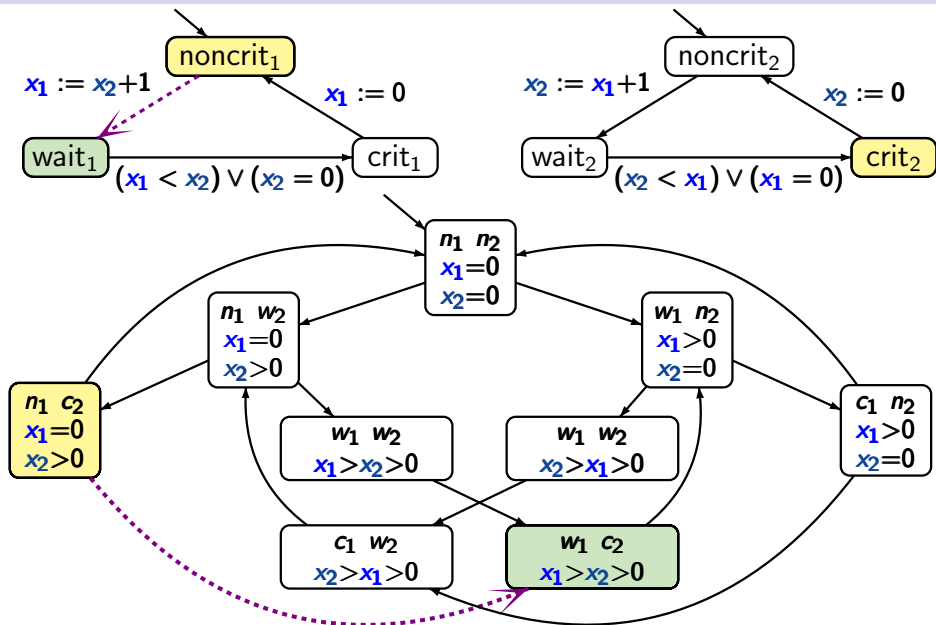
Bakery algorithm: bisimulation quotient

BSEQOR5.1-38



Bakery algorithm: bisimulation quotient

BSEQOR5.1-38



Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

bisimulation

CTL, CTL*-equivalence



computing the bisimulation quotient

abstraction stutter steps

simulation relations

CTL* state formulas

$$\Phi ::= \text{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\psi$$

CTL* path formulas

$$\psi ::= \Phi \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid \bigcirc\psi \mid \psi_1 \mathbf{U} \psi_2$$

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derived operators:

- \diamond, \square, \dots as in **LTL**

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derived operators:

- \diamond, \square, \dots as in **LTL**
- universal quantification: $\forall\psi \stackrel{\text{def}}{=} \neg\exists\neg\psi$

CTL* state formulas

$$\Phi ::= \text{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid \exists \psi$$

CTL* path formulas

$$\psi ::= \Phi \mid \psi_1 \wedge \psi_2 \mid \neg \psi \mid \bigcirc \psi \mid \psi_1 \mathbf{U} \psi_2$$

CTL: sublogic of **CTL***

- with path quantifiers \exists and \forall
- restricted syntax of **path formulas**:
 - * *no* boolean combinations of path formulas
 - * arguments of temporal operators \bigcirc and \mathbf{U} are **state formulas**

Let s_1, s_2 be states of a TS \mathcal{T} without terminal states

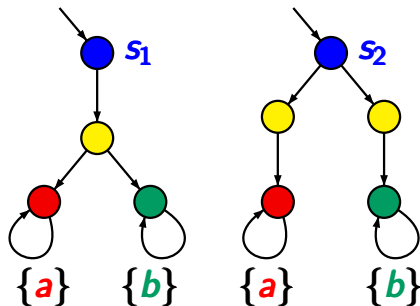
s_1, s_2 are **CTL** equivalent if for all **CTL** formulas ϕ :

$$s_1 \models \phi \quad \text{iff} \quad s_2 \models \phi$$

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s_1, s_2 are
not **CTL** equivalent

$$s_1 \models \text{EO}(\text{EO}a \wedge \text{EO}b)$$

$$s_2 \not\models \text{EO}(\text{EO}a \wedge \text{EO}b)$$

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analogous definition for **CTL*** and **LTL**

Let s_1, s_2 be states of a TS \mathcal{T} without terminal states

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s_1, s_2 are **CTL*** equivalent if for all **CTL*** formulas ϕ :

$$s_1 \models \phi \quad \text{iff} \quad s_2 \models \phi$$

s_1, s_2 are **LTL** equivalent if for all **LTL** formulas ψ :

$$s_1 \models \psi \quad \text{iff} \quad s_2 \models \psi$$

bisimulation equivalence
= **CTL** equivalence
= **CTL*** equivalence

bisimulation equivalence
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= **CTL*** equivalence

← for finite TS

bisimulation equivalence
= CTL equivalence
= CTL* equivalence

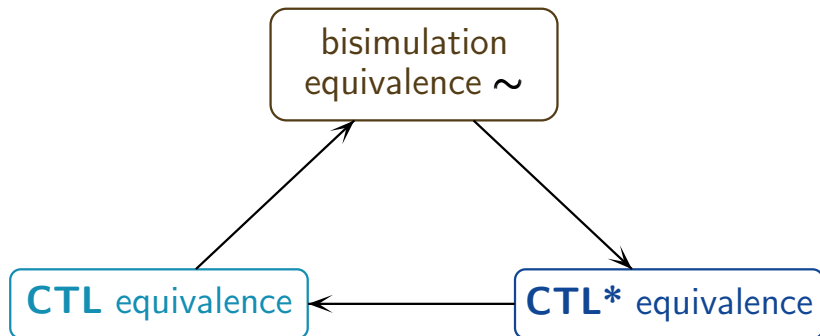
← for finite TS

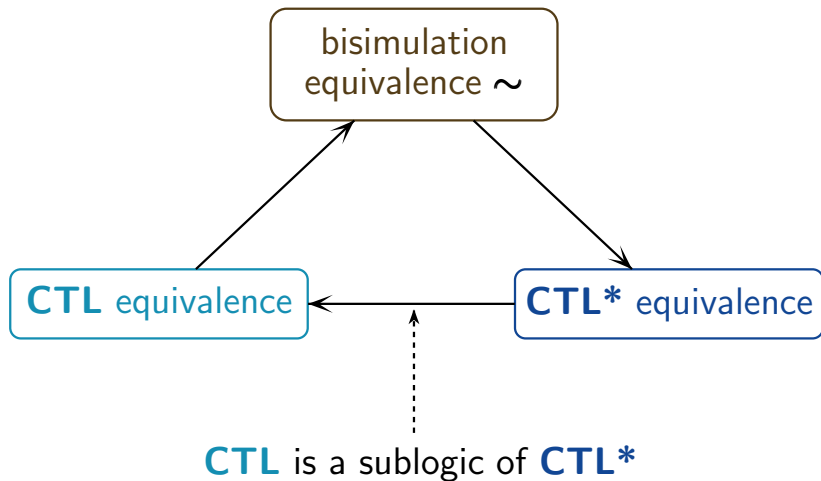
Let \mathcal{T} be a finite TS without terminal states,
and s_1, s_2 states in \mathcal{T} . Then:

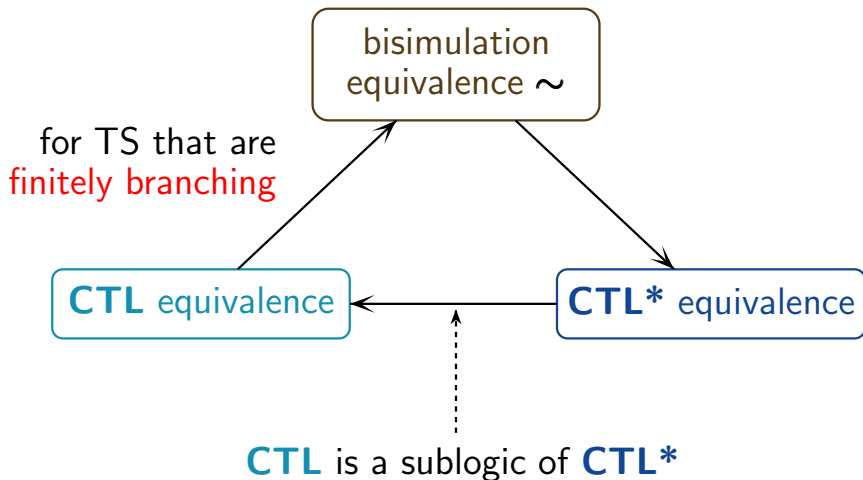
$$s_1 \sim_{\mathcal{T}} s_2$$

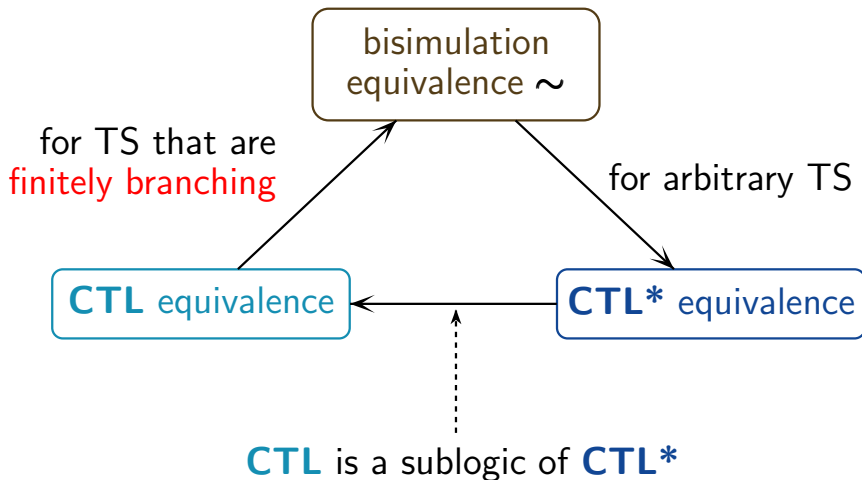
iff s_1 and s_2 are CTL equivalent

iff s_1 and s_2 are CTL* equivalent









For arbitrary (possibly infinite) transition systems without terminal states:

If s_1, s_2 are states with $s_1 \sim_{\mathcal{T}} s_2$ then for all CTL* formulas Φ :

$$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

show by structural induction on CTL* formulas:

- (a) if s_1, s_2 are states with $s_1 \sim_{\mathcal{T}} s_2$ then
for all CTL* state formulas Φ :

$$s_1 \models \Phi \text{ iff } s_2 \models \Phi$$

- (b) if π_1, π_2 are paths with $\pi_1 \sim_{\mathcal{T}} \pi_2$ then
for all CTL* path formulas φ :

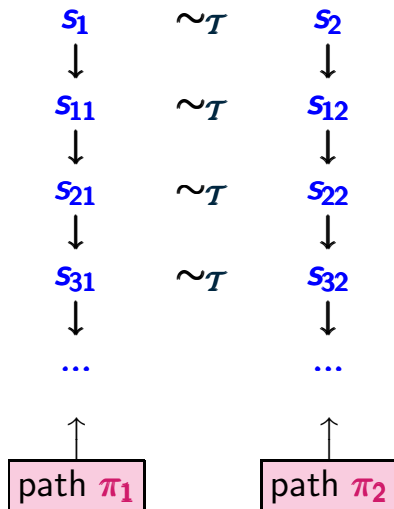
$$\pi_1 \models \varphi \text{ iff } \pi_2 \models \varphi$$

$\pi_1 \sim_{\mathcal{T}} \pi_2 \stackrel{\text{def}}{\iff} \pi_1 \text{ and } \pi_2 \text{ are statewise bisimulation equivalent}$

Bisimulation equivalence \Rightarrow CTL* equivalence

CTLEQ5.2-3

statewise bisimulation equivalent paths:



Bisimulation equivalence \Rightarrow CTL* equivalence

CTLEQ5.2-5

For all CTL* state formulas ϕ and path formulas φ :

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \phi$ iff $s_2 \models \phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

For all CTL* state formulas ϕ and path formulas φ :

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Proof by structural induction

For all CTL* state formulas Φ and path formulas φ :

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \Phi$ iff $s_2 \models \Phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Proof by structural induction

base of induction:

(a) $\Phi = \text{true}$ or $\Phi = a \in AP$

(b) $\varphi = \Phi$ for some state formula Φ
s.t. statement (a) holds for Φ

Bisimulation equivalence \Rightarrow CTL* equivalence

CTLEQ5.2-5

For all CTL* state formulas Φ and path formulas φ :

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \Phi$ iff $s_2 \models \Phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Proof by structural induction

step of induction:

(a) consider $\Phi = \Phi_1 \wedge \Phi_2, \neg\Psi$ or $\exists\varphi$ s.t.

(a) holds for Φ_1, Φ_2, Ψ

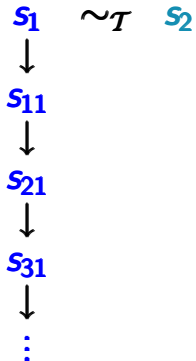
(b) holds for φ

(b) consider $\varphi = \varphi_1 \wedge \varphi_2, \neg\varphi', \bigcirc\varphi', \varphi_1 \mathbf{U} \varphi_2$ s.t.

(b) holds for $\varphi_1, \varphi_2, \varphi'$

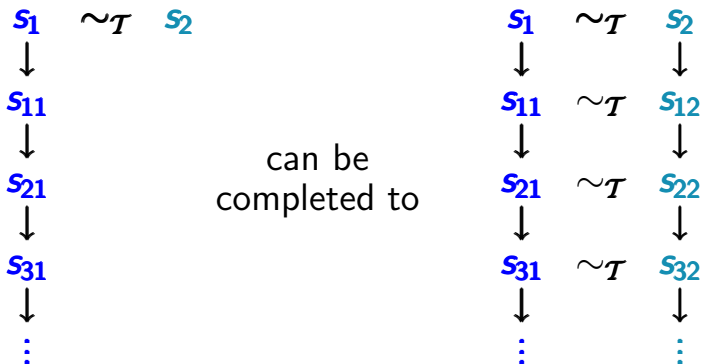
Path lifting for $\sim_{\mathcal{T}}$

CTLQ5.2-4

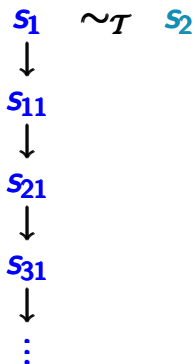


can be
completed to

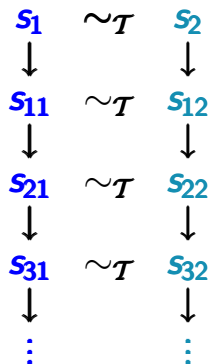




If $s_1 \sim_{\mathcal{T}} s_2$ then for all $\pi_1 \in \text{Paths}(s_1)$
 there exists $\pi_2 \in \text{Paths}(s_2)$ with $\pi_1 \sim_{\mathcal{T}} \pi_2$


 $\sim_{\mathcal{T}} s_2$

can be
completed to

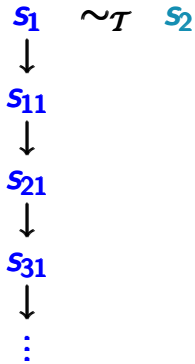

 $\sim_{\mathcal{T}}$
 s_2
 $\sim_{\mathcal{T}}$
 s_{12}
 $\sim_{\mathcal{T}}$
 s_{22}
 $\sim_{\mathcal{T}}$
 s_{32}
 \vdots

path π_1

If $s_1 \sim_{\mathcal{T}} s_2$ then for all $\pi_1 \in Paths(s_1)$
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Path lifting for \sim_T

CTLEQ5.2-4



path π_1

can be
completed to



path π_2

If $s_1 \sim_T s_2$ then for all $\pi_1 \in Paths(s_1)$
there exists $\pi_2 \in Paths(s_2)$ with $\pi_1 \sim_T \pi_2$

Correct or wrong?

CTLEQ5.2-6

If s_1, s_2 are not CTL equivalent then there exists a CTL formula ϕ with $s_1 \models \phi$ and $s_2 \not\models \phi$

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correct.

If s_1, s_2 are not **CTL** equivalent then there exists a **CTL** formula Φ with $s_1 \models \Phi$ and $s_2 \not\models \Phi$

correct.

If s_1, s_2 not **CTL** equivalent then there exists a **CTL** formula Φ with

$$s_1 \models \Phi \wedge s_2 \not\models \Phi$$

or $s_1 \not\models \Phi \wedge s_2 \models \Phi$

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correct.

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$$s_1 \models \Phi \wedge s_2 \not\models \Phi$$

or $s_1 \not\models \Phi \wedge s_2 \models \Phi \implies s_1 \models \neg\Phi \wedge s_2 \not\models \neg\Phi$

CTL equivalence \implies bisimulation equivalence

CTLEQ5.2-7A

If \mathcal{T} is a **finite** TS then, for all states s_1, s_2 in \mathcal{T} :
if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

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Proof: show that

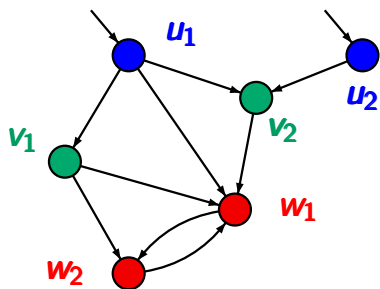
$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1, s_2 \text{ satisfy the same CTL formulas} \}$

is a bisimulation, i.e., for all $(s_1, s_2) \in \mathcal{R}$:

- (1) $L(s_1) = L(s_2)$
- (2) if $s_1 \rightarrow t_1$ then there exists a transition $s_2 \rightarrow t_2$
s.t. $(t_1, t_2) \in \mathcal{R}$

Example: CTL master formulas

CTLEQ5.2-7



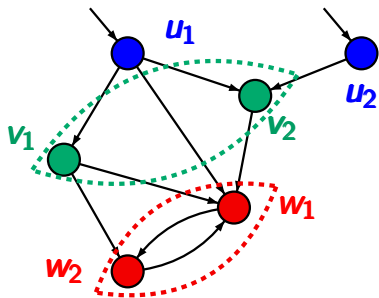
\bullet $\hat{=} \{a\}$

\bullet $\hat{=} \{b\}$

\bullet $\hat{=} \emptyset$

Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
 $= \{ (v_1, v_2), (w_1, w_2), \dots \}$

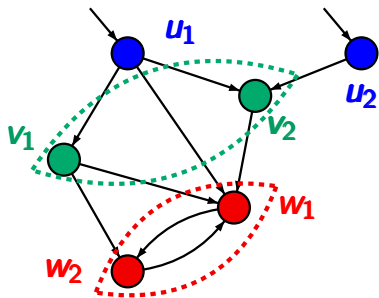
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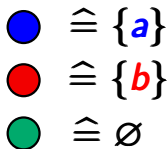
Example: CTL master formulas

CTLEQ5.2-7



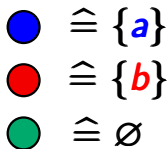
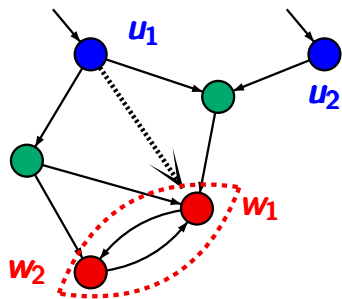
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but $u_1 \not\sim_{\mathcal{T}} u_2$



Example: CTL master formulas

CTLEQ5.2-7



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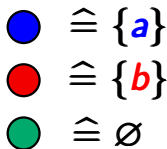
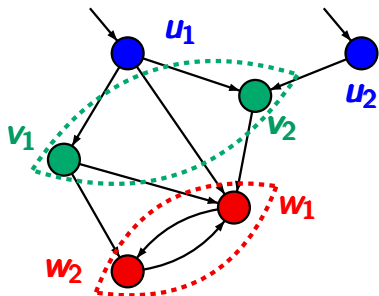
but $u_1 \not\sim_{\mathcal{T}} u_2$

as $u_1 \rightarrow \{w_1, w_2\}$

$u_2 \not\rightarrow \{w_1, w_2\}$

Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
 $= \{(v_1, v_2), (w_1, w_2), \dots\}$

CTL master formulas:

$w_1, w_2 \models ?$

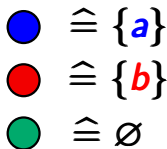
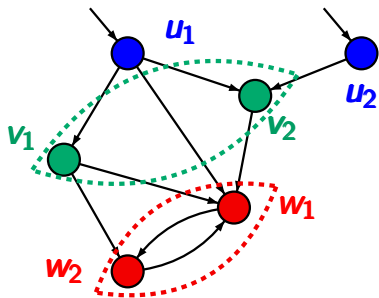
$v_1, v_2 \models ?$

$u_1 \models ?$

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Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
 $= \{(v_1, v_2), (w_1, w_2), \dots\}$

CTL master formulas:

$w_1, w_2 \models b$

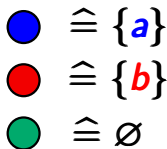
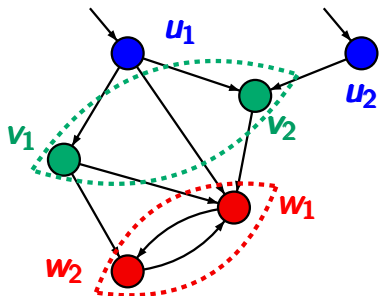
$v_1, v_2 \models ?$

$u_1 \models ?$

$u_2 \models ?$

Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
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CTL master formulas:

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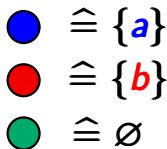
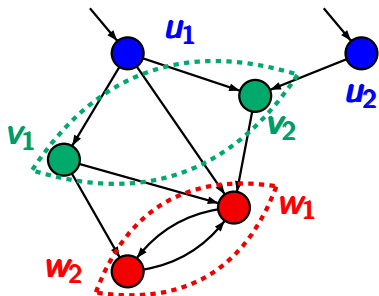
$$v_1, v_2 \models \neg a \wedge \neg b$$

$$u_1 \models ?$$

$$u_2 \models ?$$

Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
 $= \{(v_1, v_2), (w_1, w_2), \dots\}$

CTL master formulas:

$$w_1, w_2 \models b$$

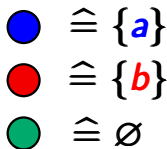
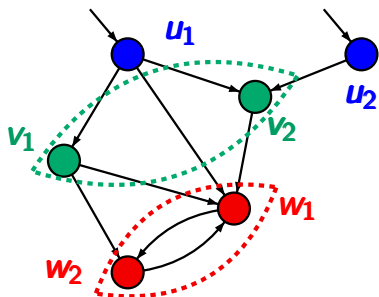
$$v_1, v_2 \models \neg a \wedge \neg b$$

$$u_1 \models (\exists \bigcirc b) \wedge a$$

$$u_2 \models ?$$

Example: CTL master formulas

CTLEQ5.2-7



bisimulation equivalence $\sim_{\mathcal{T}}$
 $= \{(v_1, v_2), (w_1, w_2), \dots\}$

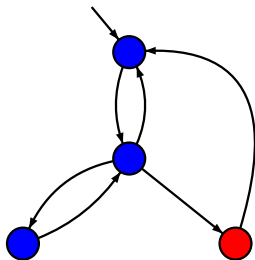
CTL master formulas:

$$w_1, w_2 \models b$$

$$v_1, v_2 \models \neg a \wedge \neg b$$

$$u_1 \models (\exists O b) \wedge a$$

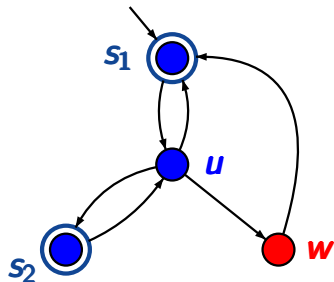
$$u_2 \models (\neg \exists O b) \wedge a$$



$$AP = \{ \textit{blue}, \textit{red} \}$$

...master formulas for $\sim_{\mathcal{T}}$ -classes?

CTLEQ5.2-8

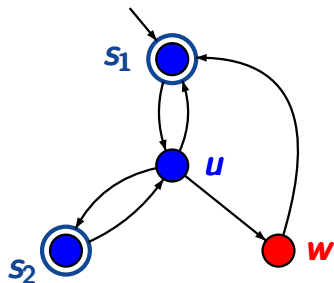


$$AP = \{ \text{blue}, \text{red} \}$$

$$s_1 \sim_{\mathcal{T}} s_2 \not\sim_{\mathcal{T}} u$$

...master formulas for \sim_T -classes?

CTLEQ5.2-8



$$AP = \{blue, red\}$$

$$s_1 \sim_T s_2 \not\sim_T u$$

$$\Phi_w = ?$$

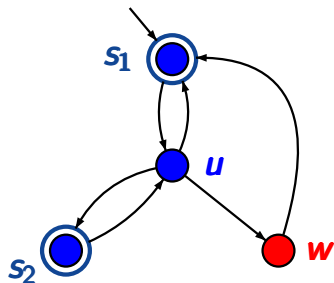
$$\Phi_C = ?$$

$$\Phi_u = ?$$

$$\text{where } C = \{s_1, s_2\}$$

...master formulas for \sim_T -classes?

CTLEQ5.2-8



$$AP = \{ \text{blue}, \text{red} \}$$

$$s_1 \sim_T s_2 \not\sim_T u$$

$$\Phi_w = \text{red}$$

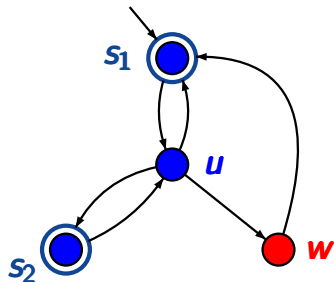
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...master formulas for $\sim_{\mathcal{T}}$ -classes?

CTLEQ5.2-8



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$$s_1 \sim_{\mathcal{T}} s_2 \not\sim_{\mathcal{T}} u$$

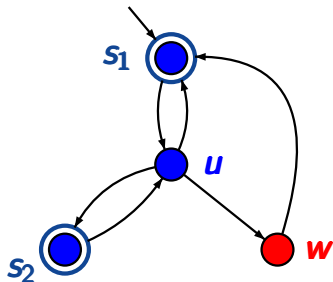
$$\Phi_w = red$$

$$\Phi_C = blue \wedge \forall O blue \quad \text{where } C = \{s_1, s_2\}$$

$$\Phi_u = ?$$

...master formulas for $\sim_{\mathcal{T}}$ -classes?

CTLEQ5.2-8



$$AP = \{blue, red\}$$

$$s_1 \sim_{\mathcal{T}} s_2 \not\sim_{\mathcal{T}} u$$

$$\Phi_w = red$$

$$\Phi_C = blue \wedge \forall O blue \quad \text{where } C = \{s_1, s_2\}$$

$$\Phi_u = \exists O red$$

CTL equivalence \implies bisimulation equivalence

If \mathcal{T} is a **finite** TS then, for all states s_1, s_2 in \mathcal{T} :
if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

- wrong for **infinite** TS
- but also holds for **finitely branching** TS

possibly infinite-state TS such that

- * the number of **initial states** is **finite**
- * for each state the number of **successors** is **finite**

CTL equivalence \implies bisimulation equivalence

CTLEQ5.2-7C

Let $\mathcal{T} = (\mathcal{S}, Act, \rightarrow, \mathcal{S}_0, AP, L)$ be **finitely branching**.

CTL equivalence \implies bisimulation equivalence

CTLEQ5.2-7c

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be **finitely branching**.

- 
- * S_0 is finite
 - * $Post(s)$ is finite for all $s \in S$

CTL equivalence \implies bisimulation equivalence

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be finitely branching.

- * S_0 is finite
- * $Post(s)$ is finite for all $s \in S$

Then, for all states s_1, s_2 in \mathcal{T} :

if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

CTL equivalence \implies bisimulation equivalence

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be finitely branching.

- * S_0 is finite
- * $Post(s)$ is finite for all $s \in S$

Then, for all states s_1, s_2 in \mathcal{T} :

if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

Proof: as for finite TS.

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CTLEQ5.2-7c

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$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1, s_2 \text{ satisfy the same } \mathbf{CTL} \text{ formulas} \}$

is a bisimulation.

CTL equivalence \implies bisimulation equivalence

CTLEQ5.2-7D

If \mathcal{T} is a **finitely branching** TS then for all states s_1, s_2 :
if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

Proof: show that

$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1, s_2 \text{ satisfy the same CTL formulas} \}$

is a bisimulation, i.e., for $(s_1, s_2) \in \mathcal{R}$:

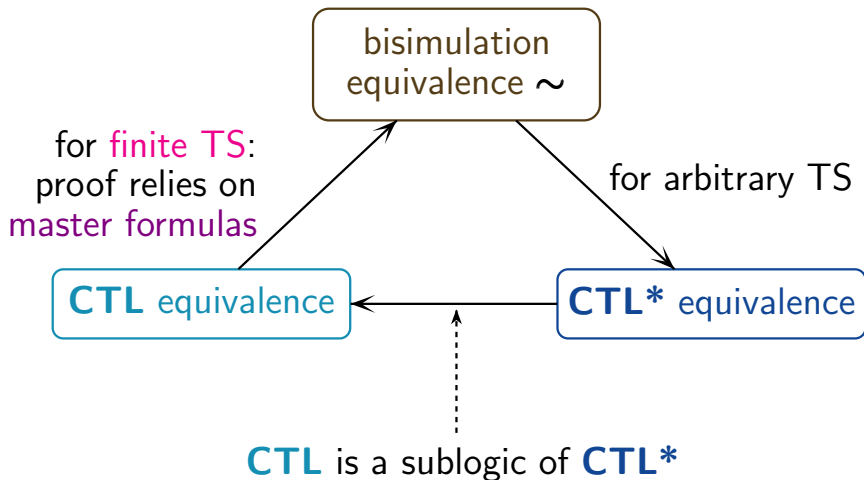
- (1) $L(s_1) = L(s_2)$
- (2) if $s_1 \rightarrow t_1$ then there exists a transition $s_2 \rightarrow t_2$
s.t. $(t_1, t_2) \in \mathcal{R}$

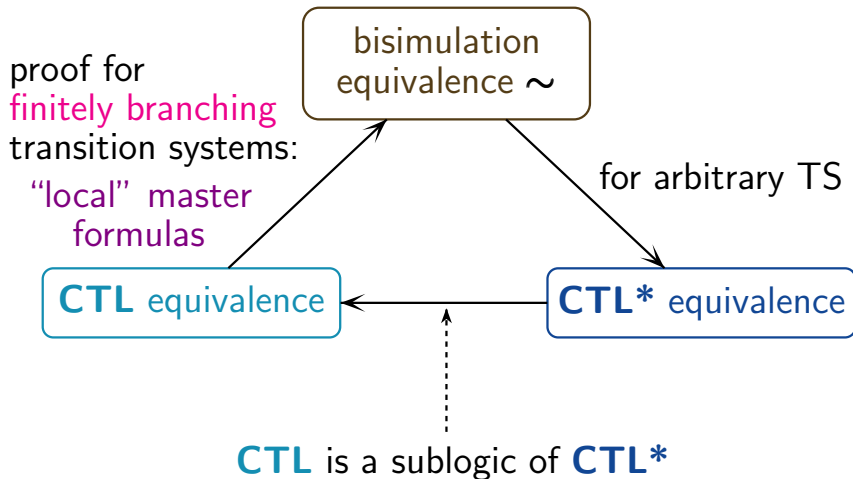
Let \mathcal{T} be a **finitely branching** TS without terminal states, and s_1, s_2 states in \mathcal{T} . Then:

$$s_1 \sim_{\mathcal{T}} s_2$$

iff s_1 and s_2 are **CTL** equivalent

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so far: we considered

- **CTL/CTL*** equivalence
- bisimulation equivalence $\sim_{\mathcal{T}}$

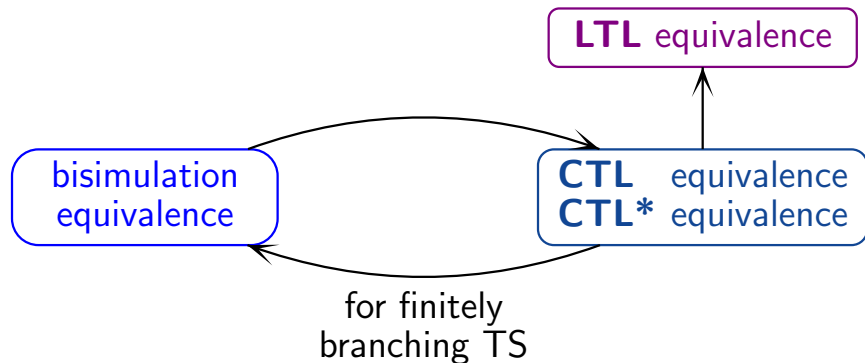
for the **states** of a single transition system \mathcal{T}

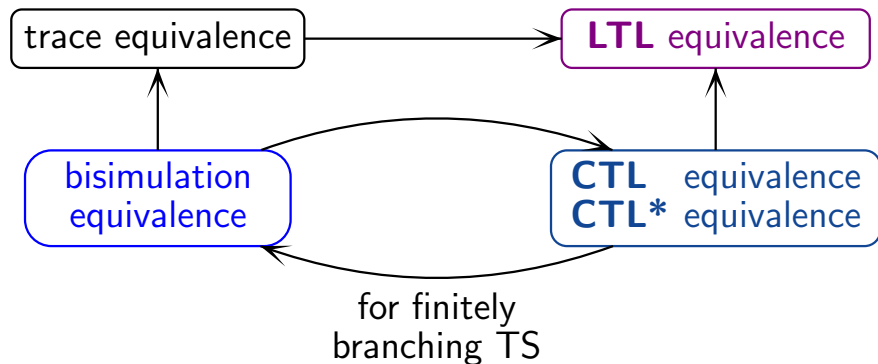
If \mathcal{T}_1 , \mathcal{T}_2 are finitely branching TS over AP without terminal states then:

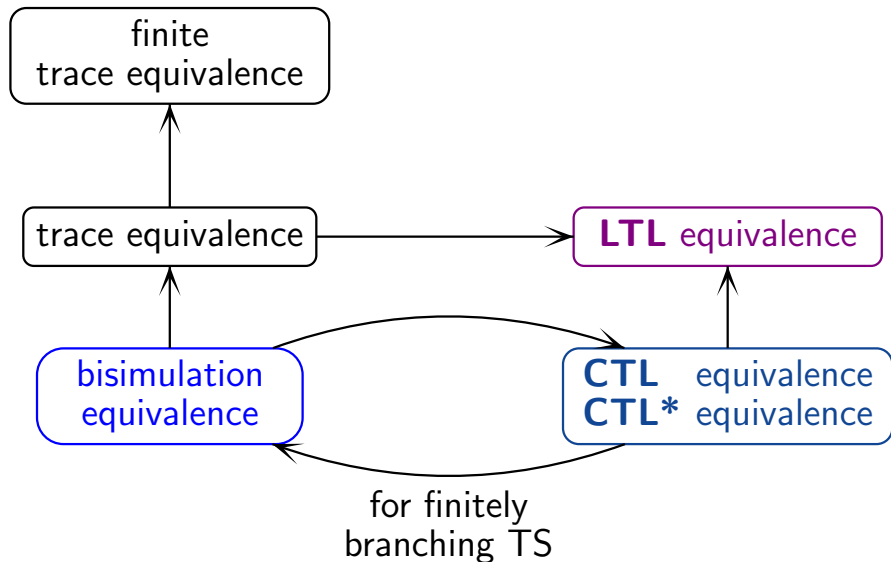
$$\mathcal{T}_1 \sim \mathcal{T}_2$$

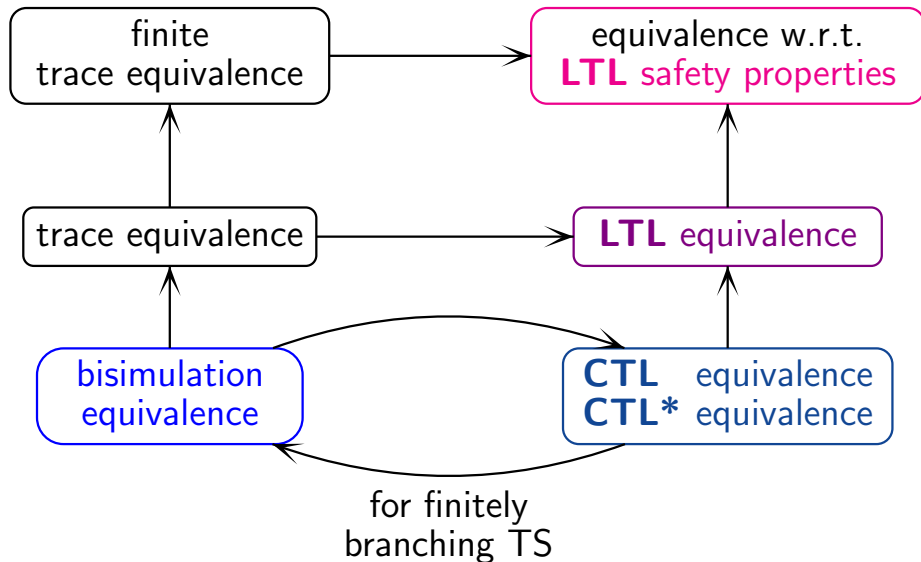
iff \mathcal{T}_1 and \mathcal{T}_2 satisfy the same **CTL** formulas

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Correct or wrong?

CTLEQ5.2-11

Let \mathcal{T} be a finite TS without terminal states and s_1, s_2 states of \mathcal{T} .

If s_1, s_2 satisfy the same $\text{CTL} \setminus \text{U}$ formulas then
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correct. see the proof

“**CTL** equivalence \implies bisimulation equivalence”

CTL_{\U}-equivalence \Rightarrow bisimulation equivalence CTLEQ5.2-11

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Proof. Show that CTL_{\U} equivalence is a bisimulation

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and $\text{Sat}(\Phi_{C,D}) \subseteq C \setminus D$

Let \mathcal{T} be a finite TS without terminal states.

\mathcal{T} and its bisimulation quotient \mathcal{T}/\sim satisfy the same **CTL*** formulas.

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correct. Recall that $\mathcal{T} \sim \mathcal{T}/\sim$ as

$$\mathcal{R} = \{(s, [s]) : s \in S\}$$

is a bisimulation for $(\mathcal{T}, \mathcal{T}/\sim)$

here: $[s] = \sim_{\mathcal{T}}$ -equivalence class of state s