Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

December 15, 2015

Overview

Continuous-time Markov chain

Continuous-time Markov chain

A CTMC is a tuple $(S, \mathbf{P}, r, \iota_{\text{init}}, AP, L)$ where

- $(S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ is a DTMC, and
- $r : S \to \mathbb{R}_{>0}$, the exit-rate function

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- residence time in state s is exponentially distributed with rate $r(s)$.
- phrased alternatively, the average residence time of state s is $\frac{1}{r(s)}$.

CTMC semantics

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Enabledness

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Residence time distribution

The probability to *take some* outgoing transition from s in $[0, t]$ is:

$$\int_0^t r(s) \cdot e^{-r(s) \cdot x} \, dx \ = \ 1 - e^{-r(s) \cdot t}$$

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- Events := sets of interval-timed paths starting in s
- Basic events := cylinder sets
- Cylinder set of finite interval-timed paths := set of all infinite timed paths with a prefix in the finite interval-timed path

Probability measure on CTMCs

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Let $s_0, \ldots, s_k \in S$ with $\mathbf{P}(s_i, s_{i+1}) > 0$ for $0 \leqslant i < k$ and $I_0, \ldots, I_{k-1}$ non-empty intervals in $\mathbb{R}_{\geqslant 0}$ with rational bounds.

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$$\begin{aligned} Cyl(s_0, I_0, \ldots, I_{k-1}, s_k) \quad = \quad \{\, \pi \in Paths(\mathcal{C}) \mid \forall 0 \leqslant i \leqslant k.\, \pi[i] = s_i \\ \text{and } i < k \;\Rightarrow\; \pi\langle i\rangle \in I_i \,\} \end{aligned}$$

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The cylinder set spanned by $s_0, I_0, \ldots, I_{k-1}, s_k$ thus consists of all infinite timed paths that have a prefix $\hat{\pi}$ that lies in $s_0, I_0, \ldots, I_{k-1}, s_k$.

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σ-algebra of a CTMC

The σ-algebra associated with CTMC $\mathcal{C}$ is the smallest σ-algebra $\mathcal{F}(Paths(s_0))$ that contains all cylinder sets $Cyl(s_0, I_0, \ldots, I_{k-1}, s_k)$ where $s_0 \ldots s_k$ is a path in the state graph of $\mathcal{C}$ (starting in s_0) and $I_0, \ldots, I_{k-1}$ range over all sequences of non-empty intervals in $\mathbb{R}_{\geqslant 0}$.

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Solving the integral

$$Pr\big(Cyl(s_0, I_0, \ldots, I_{k-2}, s_{k-1})\big) \cdot \mathbf{P}(s_{k-1}, s_k) \cdot \left(e^{-r(s_{k-1}) \cdot \inf I_{k-1}} - e^{-r(s_{k-1}) \cdot \sup I_{k-1}} \right).$$

Zeno theorem

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Zeno path

Path $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} s_3 \ldots\ldots$ is called Zeno [1] if $\sum_i t_i$ converges.

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Thanks to the following theorem, Zeno paths do not harm for CTMCs.

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For all states s in any CTMC, $Pr\{\, \pi \in Paths(s) \mid \pi \text{ is Zeno} \,\} = 0$.

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Proof:

On the blackboard.

Overview

1. Recall: continuous-time Markov chains
2. Probability measure on CTMC paths
3. Reachability probabilities
   - Untimed reachability
   - Timed reachability
   - Reduction to transient analysis
   - Bisimulation and timed reachability
4. Summary

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Proof:

Consider $\Diamond G$. $\Diamond G$ is the union of all cylinders $Cyl(s_0, [0, \infty), \ldots, [0, \infty), s_n)$ where $s_0, \ldots, s_{n-1} \notin G$ and $s_n \in G$.

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$$x_s = \underbrace{\sum_{t \in S \setminus G} \mathbf{P}(s,t) \cdot x_t}_{\text{reach } G \text{ via } t \in S \setminus G} + \underbrace{\sum_{u \in G} \mathbf{P}(s,u)}_{\text{reach } G \text{ in one step}}$$

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Justification:

As the above temporal logic formulas or events do not refer to elapsed time, it is not surprising that they can be checked on the embedded DTMC.

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$$x_s(t) = \int_0^t \sum_{s' \in S} \underbrace{\mathbf{R}(s,s') \cdot e^{-r(s) \cdot x}}_{\substack{\text{probability to move to} \\ \text{state } s' \text{ at time } x}} \cdot \underbrace{x_{s'}(t-x)}_{\substack{\text{prob. to fulfill} \\ \Diamond^{\leqslant t-x} G \text{ from } s'}} \, dx$$

Timed reachability probabilities: example

On the blackboard.

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Let $\mathcal{C}$ be a CTMC with finite state space S, $s \in S$, $t \in \mathbb{R}_{\geqslant 0}$ and $G, F \subseteq S$.

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$$x_s(t) \;=\; \int_0^t \sum_{s' \in S} \underbrace{\mathbf{R}(s, s') \cdot e^{-r(s) \cdot x}}_{\text{probability to move to state } s' \text{ at time } x} \cdot \underbrace{x_{s'}(t-x)}_{\text{prob. to fulfill } \overline{F} \, \mathsf{U}^{\leqslant t-x} \, G \text{ from } s'} \; dx$$

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provided F and G are closed under $\sim_m$ and $\approx_m$, respectively.

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Proof:

Left as an exercise.

Example

Overview

1. Recall: continuous-time Markov chains
2. Probability measure on CTMC paths
3. Reachability probabilities
   - Untimed reachability
   - Timed reachability
   - Reduction to transient analysis
   - Bisimulation and timed reachability
4. Summary

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- Weak and strong bisimulation preserve timed reachability probabilities.