Modeling and Verification of Probabilistic Systems

Joost-Pieter Katoen

Lehrstuhl für Informatik 2
Software Modeling and Verification Group

http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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Overview

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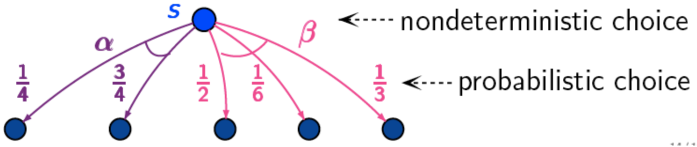

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The path

$$\pi \;=\; s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$$

is called a $\mathfrak{S}$-path if $\alpha_i = \mathfrak{S}(s_0 \ldots s_{i-1})$ for all $i > 0$.

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$\mathcal{M}_{\mathfrak{S}}$ is infinite, even if the MDP $\mathcal{M}$ is finite. Since policy $\mathfrak{S}$ might select different actions for finite paths that end in the same state s, a policy as defined above is also referred to as *history-dependent*.

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$$Pr^{\mathfrak{S}}(P) \;=\; Pr^{\mathcal{M}_{\mathfrak{S}}}(P) \;=\; Pr_{\mathcal{M}_{\mathfrak{S}}}\{\, \pi \in Paths(\mathcal{M}_{\mathfrak{S}}) \mid trace(\pi) \in P \,\}.$$

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Policy $\mathfrak{S}$ is positional if it always selects the same action in a given state. This choice is independent of what has happened in the history, i.e., which path led to the current state.

Finite-memory policies

- *Finite-memory policies* (shortly: fm-policies) are a generalisation of positional policies.
- The behavior of an fm-policy is described by a deterministic finite automaton (DFA).
- The selection of the action to be performed in the MDP $\mathcal{M}$ depends on the current state of $\mathcal{M}$ (as before) and the current state (called *mode*) of the policy, i.e., the DFA.

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A *finite-memory policy* $\mathfrak{S}$ for $\mathcal{M}$ is a tuple $\mathfrak{S} = (Q, act, \Delta, start)$ with:

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- $start : S \rightarrow Q$ is a function that selects a starting mode for state $s \in S$.

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- The policy changes to mode $\Delta(q, s)$, while $\mathcal{M}$ performs the selected action α and randomly moves to the next state according to the distribution $\mathbf{P}(s, \alpha, \cdot)$.

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Positional policies can be considered as fm-policies with just a single mode.

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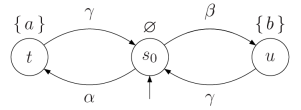

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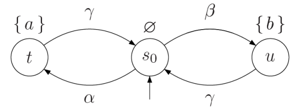

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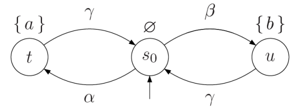

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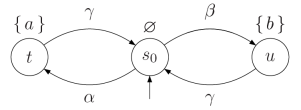

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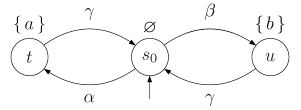

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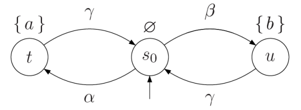

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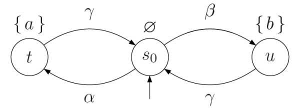

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$$Pr_{\mathfrak{S}_\alpha}(s_0 \models \Diamond a \wedge \Diamond b) \;=\; Pr_{\mathfrak{S}_\beta}(s_0 \models \Diamond a \wedge \Diamond b) \;=\; 0.$$

Now consider fm-policy $\mathfrak{S}_{\alpha\beta}$ which alternates between selecting α and β.
Then: $Pr_{\mathfrak{S}_{\alpha\beta}}(s_0 \models \Diamond a \wedge \Diamond b) \;=\; 1.$

Thus, the class of positional policies is insufficiently powerful to characterise minimal (or maximal) probabilities for ω-regular properties.

Other kinds of policies

- Counting policies that base their decision on the number of visits to a state, or the length of the history (i.e., number of visits to all states)

- Partial-observation policies that base their decision on the trace $L(s_0) \ldots L(s_n)$ of the history $s_0 \ldots s_n$.

- Randomised policies. This is applicable to all (deterministic) policies.

  For instance, a randomised positional policy $\mathfrak{S} : S \rightarrow Dist(Act)$, where $Dist(X)$ is the set of probability distributions on X, such that $\mathfrak{S}(s)(\alpha) > 0$ iff $\alpha \in Act(s)$. Similar can be done for fm-policies and history-dependent policies etc..

- There is a strict hierarchy of policies, showing their expressiveness (black board).

Overview

1. Markov Decision Processes
2. Policies
   - Positional policies
   - Finite-memory policies
3. Reachability probabilities
   - Mathematical characterisation
   - Value iteration
   - Linear programming
   - Policy iteration
4. Summary

Reachability probabilities

Reachability probabilities

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Reachability probabilities

Let $\mathcal{M}$ be an MDP with state space S and $\mathfrak{S}$ be a policy on $\mathcal{M}$. The reachability probability of $G \subseteq S$ from state $s \in S$ under policy $\mathfrak{S}$ is:

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Maximal and minimal reachability probabilities

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In a similar way, the maximal reachability probability of $G \subseteq S$ is:

$$Pr^{\max}(s \models \lozenge G) \;=\; \sup_{\mathfrak{S}} Pr^{\mathfrak{S}}(s \models \lozenge G).$$

where policy $\mathfrak{S}$ ranges over all, infinitely (countably) many, policies.

Examples

Maximal reachability probabilities

MInimal guarantees for safety properties

Reasoning about the maximal probabilities for $\Diamond G$ is needed, e.g., for showing that $Pr^{\mathfrak{S}}(s \models \Diamond G) \leqslant \varepsilon$ for all policies $\mathfrak{S}$ and some small upper bound $0 < \varepsilon \leqslant 1$.

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$$Pr^{\mathfrak{S}}(s \models \Box \neg G) \;\geqslant\; 1 - \varepsilon \quad \text{for all policies } \mathfrak{S}.$$

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$$Pr^{\mathfrak{S}}(s \models \Box \neg G) \;\geqslant\; 1 - \varepsilon \quad \text{for all policies } \mathfrak{S}.$$

The task to compute $Pr^{\max}(s \models \Diamond G)$ can thus be understood as showing that a safety property (namely $\Box \neg G$) holds with sufficiently large probability, viz. $1 - \varepsilon$, regardless of the resolution of nondeterminism.

Equation system for max-reach probabilities

[1]Richard Bellman, an american mathematician (1920–1984), also known from the Bellman-Form shortest path algorithm.

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- If $s \in G$, then $x_s = 1$.

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This is a Bellman [1] equation as used in dynamic programming.

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Example

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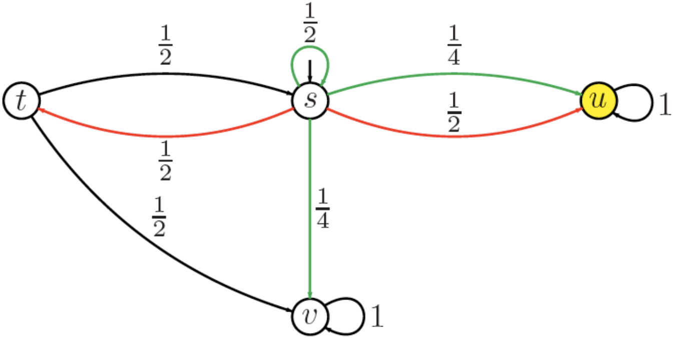

equation system for reachability objective $\Diamond\{ u \}$ is:

$x_u = 1$ and $x_v = 0$

$$x_s \; = \; \max\{ \tfrac{1}{2}x_s + \tfrac{1}{4}x_u + \tfrac{1}{4}x_v, \tfrac{1}{2}x_u + \tfrac{1}{2}x_t \} \quad \text{and} \quad x_t = \tfrac{1}{2}x_s + \tfrac{1}{2}x_v$$

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The previous theorem suggests to calculate the values

$$x_s = Pr^{\max}(s \models \Diamond G)$$

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Note that $x_s^{(0)} \leqslant x_s^{(1)} \leqslant x_s^{(2)} \leqslant \ldots$.

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Note that $x_s^{(0)} \leqslant x_s^{(1)} \leqslant x_s^{(2)} \leqslant \ldots$. Thus, the values $Pr^{\max}(s \models \lozenge G)$ can be approximated by successively computing the vectors

$$(x_s^{(0)}), (x_s^{(1)}), (x_s^{(2)}), \ldots,$$

until $\max_{s \in S} |x_s^{(n+1)} - x_s^{(n)}|$ is below a certain (typically very small) threshold.

Positional policies suffice for reach probabilities

Positional policies suffice for reach probabilities

Existence of optimal positional policies

Let $\mathcal{M}$ be a finite MDP with state space S, and $G \subseteq S$. There exists a positional policy $\mathfrak{S}$ such that for any $s \in S$ it holds:

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Proof:

On the blackboard.

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The preprocessing required to compute the set

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$$T = \bigcup_{n \geqslant 0} T_n$$

and $T_0 = G$ and, for $n \geqslant 0$:

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It follows: $Pr^{\min}(s \models \Diamond G) > 0$ if and only if $s \in T$.

Positional policies for min-reach probabilities

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Existence of optimal positional policies

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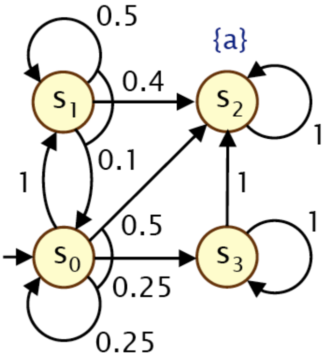
Proof:

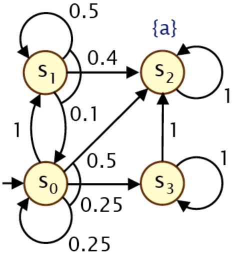
Similar to the case for maximal reachability probabilities.

Example value iteration



Determine $Pr^{\min}(s_i \models \Diamond\{ s_2 \})$.

Example value iteration



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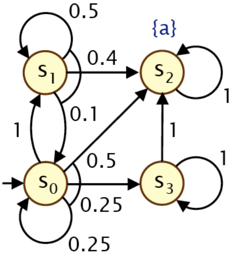

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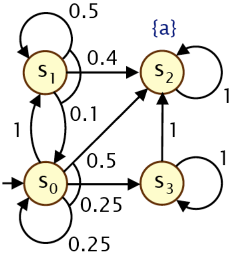

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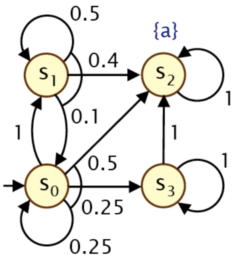

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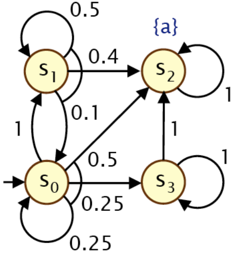
Example value iteration



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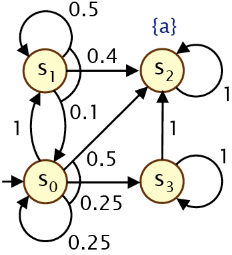
$$= (0, 0.4, 1, 0)$$

Example value iteration



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   $= (0, 0.4, 1, 0)$
4. $(x_s^{(2)}) = (\min(1{\cdot}0.4, 0.25{\cdot}0{+}0.25{\cdot}0{+}0.5{\cdot}1),$
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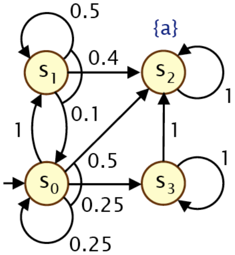

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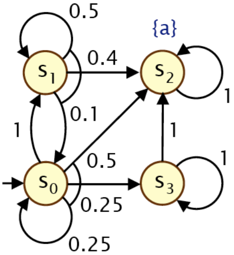

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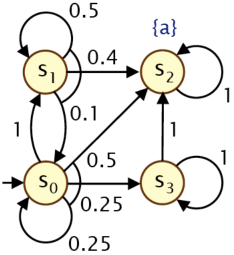
   $= (0.4, 0.6, 1, 0)$
5. $(x_s^{(3)}) = \ldots\ldots$

Example value iteration



Determine
$Pr^{\min}(s_i \models \Diamond\{ s_2 \})$

	$[x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}]$
n=0:	[0.000000, 0.000000, 1, 0]
n=1:	[0.000000, 0.400000, 1, 0]
n=2:	[0.400000, 0.600000, 1, 0]
n=3:	[0.600000, 0.740000, 1, 0]
n=4:	[0.650000, 0.830000, 1, 0]
n=5:	[0.662500, 0.880000, 1, 0]
n=6:	[0.665625, 0.906250, 1, 0]
n=7:	[0.666406, 0.919688, 1, 0]
n=8:	[0.666602, 0.926484, 1, 0]
...	
n=20:	[0.666667, 0.933332, 1, 0]
n=21:	[0.666667, 0.933332, 1, 0]
	$\approx$ [2/3, 14/15, 1, 0]

Optimal positional policy

Positional policies $\mathfrak{S}_{\text{min}}$ and $\mathfrak{S}_{\text{max}}$ thus yield:

$$
\begin{aligned}
Pr^{\mathfrak{S}_{\text{min}}}(s \models \Diamond G) &= Pr^{\text{min}}(s \models \Diamond G) \quad \text{for all states } s \in S \\
Pr^{\mathfrak{S}_{\text{max}}}(s \models \Diamond G) &= Pr^{\text{max}}(s \models \Diamond G) \quad \text{for all states } s \in S
\end{aligned}
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Optimal positional policy

Positional policies $\mathfrak{S}_{\min}$ and $\mathfrak{S}_{\max}$ thus yield:

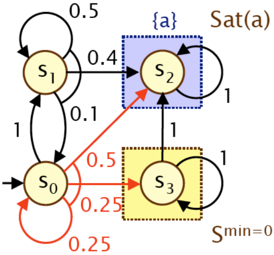
$$Pr^{\mathfrak{S}_{\min}}(s \models \lozenge G) = Pr^{\min}(s \models \lozenge G) \quad \text{for all states } s \in S$$
$$Pr^{\mathfrak{S}_{\max}}(s \models \lozenge G) = Pr^{\max}(s \models \lozenge G) \quad \text{for all states } s \in S$$

These policies are obtained as follows:

$$\mathfrak{S}_{\min}(s) = \arg\min\{ \sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot Pr^{\min}(t \models \lozenge G) \mid \alpha \in Act \}$$

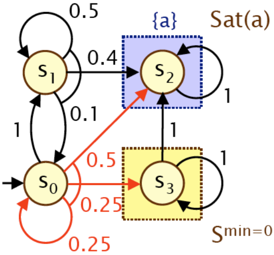
$$\mathfrak{S}_{\max}(s) = \arg\max\{ \sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot Pr^{\max}(t \models \lozenge G) \mid \alpha \in Act \}$$

Optimal positional policy

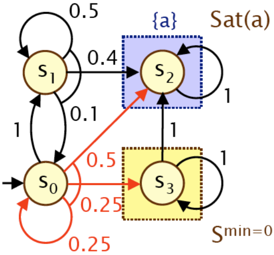


Optimal positional policy

- Outcome of the value iteration $(\, x_s \,) = (\frac{2}{3}, \frac{14}{15}, 1, 0)$



Optimal positional policy



- Outcome of the value iteration $(\, x_s \,) = (\frac{2}{3}, \frac{14}{15}, 1, 0)$
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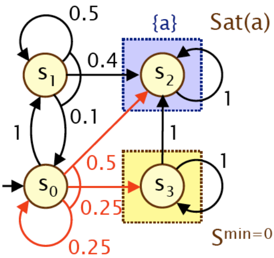
Optimal positional policy



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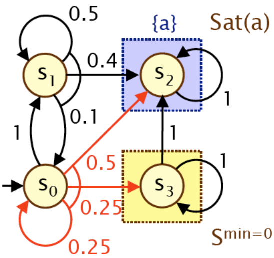

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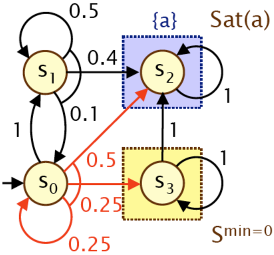

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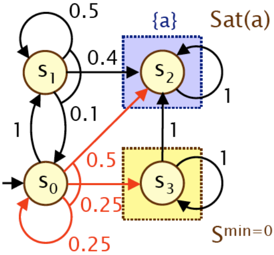

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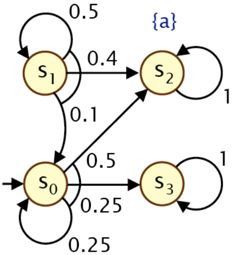
- Note that the minimal reach-probability is unique; the optimal policy need not to be unique.

Induced DTMC



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An alternative approach

A viable alternative to value iteration is linear programming.

Linear programming

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Optimisation of a linear objective function subject to linear (in)equalities.

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Let $x_1, \ldots, x_n$ be non-negative real-valued variables. Maximise (or minimise) the objective function:

$$c_1 \cdot x_1 + c_2 \cdot x_2 + \ldots + c_n \cdot x_n \quad \text{for constants } c_1, \ldots, c_n \in \mathbb{R}$$

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subject to the constraints

$$\begin{aligned} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n &\leqslant b_1 \\ \ldots \ldots & \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \ldots + a_{mn} \cdot x_n &\leqslant b_m. \end{aligned}$$

Solution techniques: e.g., Simplex, ellipsoid method, interior point method.

Maximal reach probabilities as a linear program

Maximal reach probabilities as a linear program

Linear program for max-reach probabilities

Consider a finite MDP with state space S, and $G \subseteq S$. The values $x_s = Pr^{\max}(s \models \lozenge G)$ are the unique solution of the *linear program*:

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$$x_s \geqslant \sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot x_t$$

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Proof:

See lecture notes.

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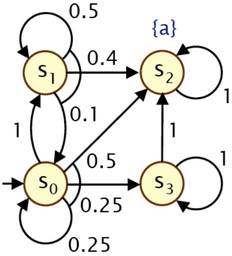

Determine

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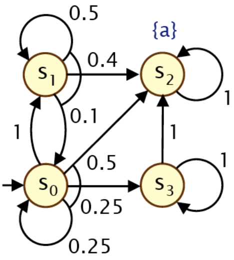

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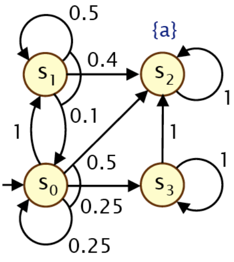

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- $G = \{ s_2 \}, S_{=0}^{\min} = \{ s_3 \}, S \setminus (G \cup S_{=0}^{\min}) = \{ s_0, s_1 \}$.
- Maximise $x_0 + x_1$ subject to the constraints:

$$\begin{aligned} x_0 &\leqslant x_1 \\ x_0 &\leqslant \tfrac{1}{4} \cdot x_0 + \tfrac{1}{2} \\ x_1 &\leqslant \tfrac{1}{10} \cdot x_0 + \tfrac{1}{2} \cdot x_1 + \tfrac{2}{5} \end{aligned}$$

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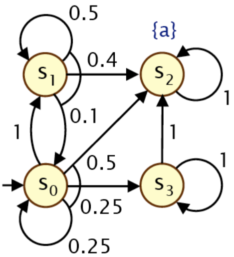

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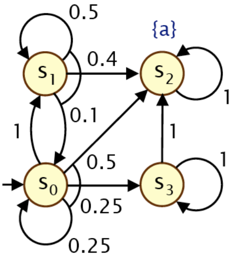

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$$
\begin{aligned}
x_0 &\leqslant x_1 \\
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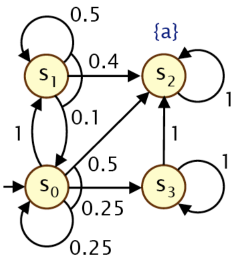
\end{aligned}
$$

Example linear programming



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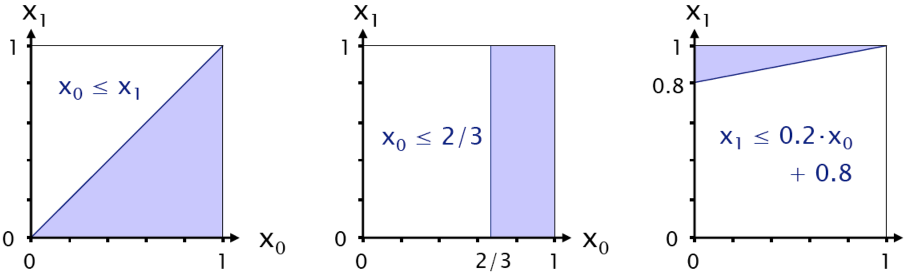

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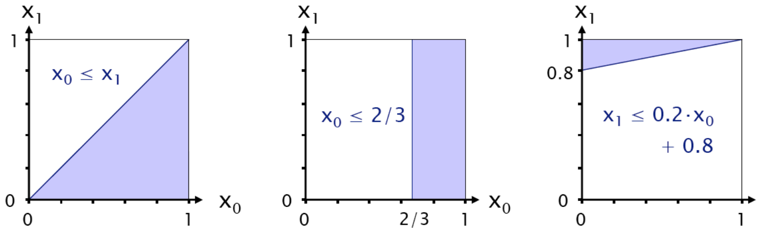

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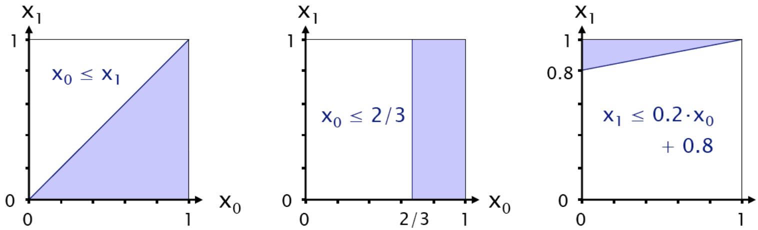
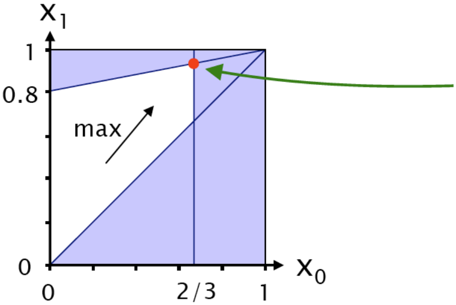




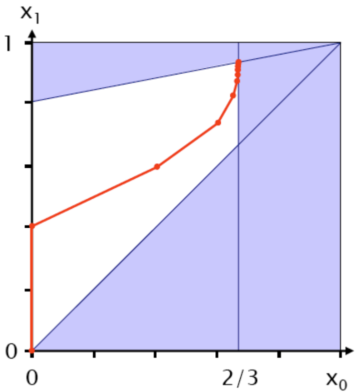
Solution:

(x_0, x_1)

$=$

$(2/3, 14/15)$

Value iteration vs. linear programming



	$[\, x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)} \,]$
n=0:	[0.000000, 0.000000, 1, 0]
n=1:	[0.000000, 0.400000, 1, 0]
n=2:	[0.400000, 0.600000, 1, 0]
n=3:	[0.600000, 0.740000, 1, 0]
n=4:	[0.650000, 0.830000, 1, 0]
n=5:	[0.662500, 0.880000, 1, 0]
n=6:	[0.665625, 0.906250, 1, 0]
n=7:	[0.666406, 0.919688, 1, 0]
n=8:	[0.666602, 0.926484, 1, 0]
...	
n=20:	[0.666667, 0.933332, 1, 0]
n=21:	[0.666667, 0.933332, 1, 0]
	≈ [2/3, 14/15, 1, 0]

This curve shows how the value iteration approach approximates the solution.

Time complexity

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For finite MDP $\mathcal{M}$ with state space S, $G \subseteq S$ and $s \in S$, the values $Pr^{\max}(s \models \Diamond G)$ can be computed in time polynomial in the size of $\mathcal{M}$. The same holds for $Pr^{\min}(s \models \Diamond G)$.

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Corollary

For finite MDPs, the question whether $Pr^{\mathfrak{S}}(s \models \Diamond G) \leqslant p$ for some rational $p \in [0, 1[$ is decidable in polynomial time.

Yet another alternative approach

A viable alternative to value iteration and linear programming is policy iteration.

Policy iteration

Value iteration

In value iteration, we iteratively attempt to improve the minimal (or maximal) reachability probabilities by starting with an underestimation, viz. zero for all states.

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Policy iteration

In policy iteration, the idea is to start with an arbitrary positional policy and improve it for each state in a step-by-step fashion, so as to determine the optimal one.

Policy iteration

Policy iteration

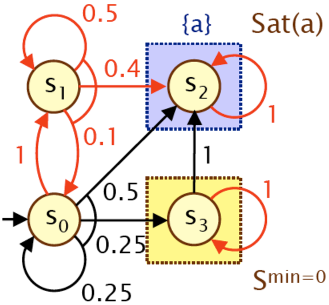
1. Start with an arbitrary positional policy $\mathfrak{S}$ that selects some $\alpha \in Act(s)$ for each state $s \in S \setminus G \cup S^{\min}_{=0}$.
2. Compute the reachability probabilities $Pr^{\mathfrak{S}}(s \models \lozenge G)$. This amounts to solving a linear equation system on DTMC $\mathcal{M}_{\mathfrak{S}}$.
3. Improve the policy in every state according to the following rules:

$$\begin{aligned}\mathfrak{S}^{(i+1)}(s) &= \arg\min\{ \sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot Pr^{\mathfrak{S}^{(i)}}(t \models \lozenge G) \mid \alpha \in Act \} \text{ or} \\ \mathfrak{S}^{(i+1)}(s) &= \arg\max\{ \sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot Pr^{\mathfrak{S}^{(i)}}(t \models \lozenge G) \mid \alpha \in Act \}\end{aligned}$$

4. Repeat steps 2. and 3. until the policy does not change.
5. Termination[2]: finite number of states and improvement of min/max probabilities each time.

[2] For a proof, see Section 6.7 of the book by Tijms on A First Course in Stochastic

Policy iteration: example



- Let $G = \{ s_2 \}$.
- Consider an arbitrary policy $\mathfrak{S}$.

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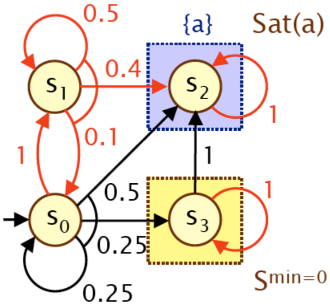

- Let $G = \{ s_2 \}$.
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- Compute $x_i = Pr^{\mathfrak{S}}(s_i \models \Diamond G)$ for all i.

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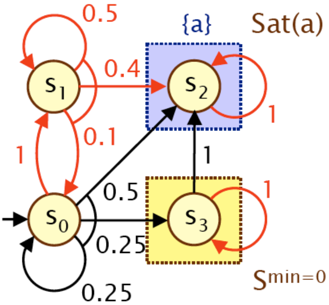

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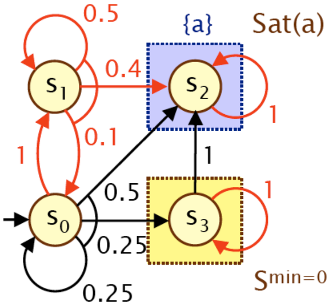

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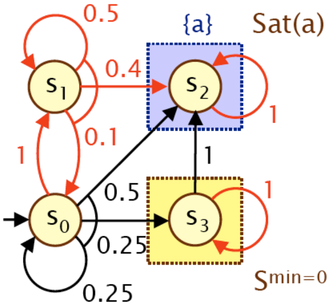
Policy iteration: example



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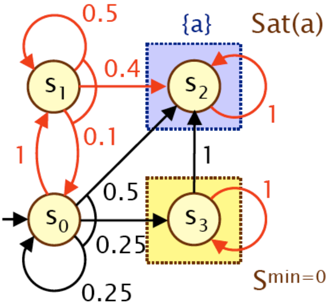

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- Change policy $\mathfrak{S}$ in s_0, yielding policy $\mathfrak{S}'$.

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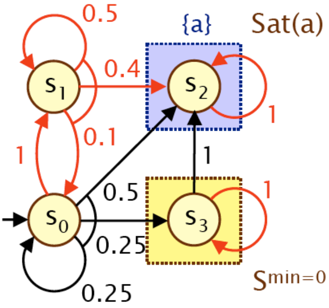

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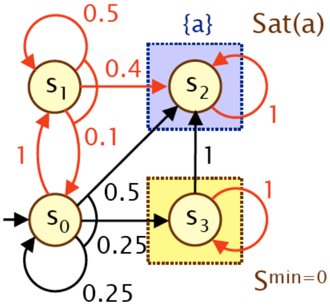
- Change policy $\mathfrak{S}$ in s_0, yielding policy $\mathfrak{S}'$.
- This yields $\min(1 \cdot 1, \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1)$

Policy iteration: example



- Let $G = \{ s_2 \}$.
- Consider an arbitrary policy $\mathfrak{S}$.
- Compute $x_i = Pr^{\mathfrak{S}}(s_i \models \Diamond G)$ for all i.
- Then: $x_2 = 1$, $x_3 = 0$,

  and $x_0 = x_1$, $x_1 = \frac{1}{10} \cdot x_0 + \frac{1}{2} \cdot x_1 + \frac{2}{5}$.
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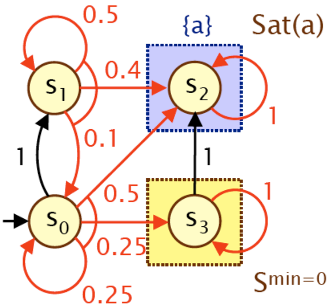
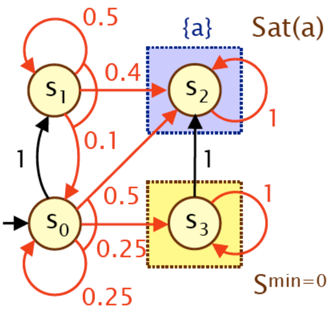
  that is, $\min(1, \frac{3}{4}) \; = \; \frac{3}{4}$.

Policy iteration: example



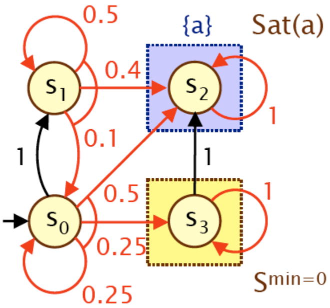
- Let $G = \{ s_2 \}$.
- Consider the adapted policy $\mathfrak{S}'$.

Policy iteration: example



- Let $G = \{ s_2 \}$.
- Consider the adapted policy $\mathfrak{S}'$.
- Compute $x_i = Pr^{\mathfrak{S}'}(s_i \models \Diamond G)$ for all i.

Policy iteration: example



- Let $G = \{ s_2 \}$.
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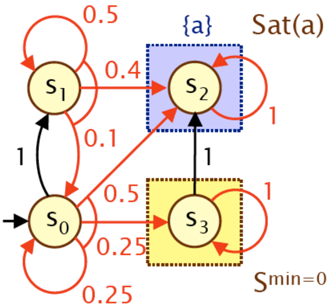
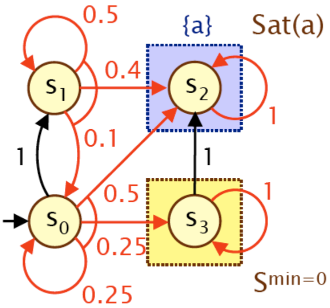


- Let $G = \{ s_2 \}$.
- Consider the adapted policy $\mathfrak{S}'$.
- Compute $x_i = Pr^{\mathfrak{S}'}(s_i \models \lozenge G)$ for all i.
- Then: $x_2 = 1$, $x_3 = 0$,

  and $x_0 = \frac{1}{4} \cdot x_0 + \frac{1}{2}$, $x_1 = \frac{1}{10} \cdot x_0 + \frac{1}{2} \cdot x_1 + \frac{2}{5}$.

Policy iteration: example



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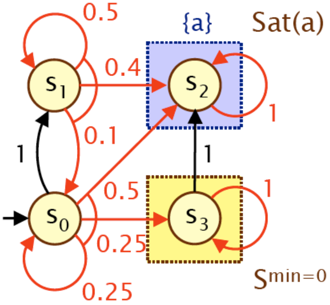
- This yields $x_0 = \frac{2}{3}$, $x_1 = \frac{14}{15}$, $x_2 = 1$ and $x_3 = 0$.

Policy iteration: example



- Let $G = \{ s_2 \}$.
- Consider the adapted policy $\mathfrak{S}'$.
- Compute $x_i = Pr^{\mathfrak{S}'}(s_i \models \Diamond G)$ for all i.
- Then: $x_2 = 1$, $x_3 = 0$,

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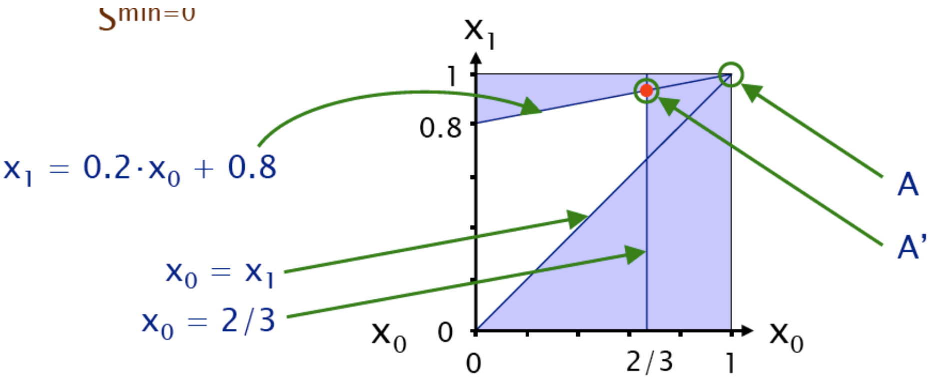
- This yields $x_0 = \frac{2}{3}$, $x_1 = \frac{14}{15}$, $x_2 = 1$ and $x_3 = 0$.
- This policy is optimal.

Graphical representation of policy iteration



where A denotes policy $\mathfrak{S}$ and A' policy $\mathfrak{S}'$.

Overview

Summary

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Important points

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