Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

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Overview

Labeled transition system

Transition system

A *(labeled) transition system* TS is a structure $(S, Act, \longrightarrow, I_0, AP, L)$ where

- S is a (possibly infinitely countable) set of states.
- Act is a (possibly infinitely countable) set of actions.
- $\longrightarrow \subseteq S \times Act \times S$ is a transition relation.
- $I_0 \subseteq S$ the set of initial states.
- AP is a set of atomic propositions.
- $L : S \rightarrow 2^{AP}$ is the labeling function.

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Notation

We write $s \xrightarrow{\alpha} s'$ instead of $(s, \alpha, s') \in \longrightarrow$.

Strong bisimulation

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Strong bisimulation relation [Milner, 1980 & Park, 1981]

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Not every bisimulation relation is transitive.

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Remarks

Not every bisimulation relation is transitive. But: $\sim$ is an equivalence.

Strong bisimulation

Pictorial representation

$$\begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & \\ t & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & R \\ t & \xrightarrow{\alpha} & t' \end{array}$$

Strong bisimulation

Pictorial representation

s	$\xrightarrow{\alpha}$	s'		s	$\xrightarrow{\alpha}$	s'
R			can be completed to	R		R
t				t	$\xrightarrow{\alpha}$	t'

and

s				s	$\xrightarrow{\alpha}$	s'
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t	$\xrightarrow{\alpha}$	t'		t	$\xrightarrow{\alpha}$	t'

Strongly bisimilar transition systems

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Let TS_1, TS_2 be transition systems over the same set of atomic propositions with initial states $I_{0,1}$ and $I_{0,2}$, respectively.

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Let TS_1, TS_2 be transition systems over the same set of atomic propositions with initial states $I_{0,1}$ and $I_{0,2}$, respectively.

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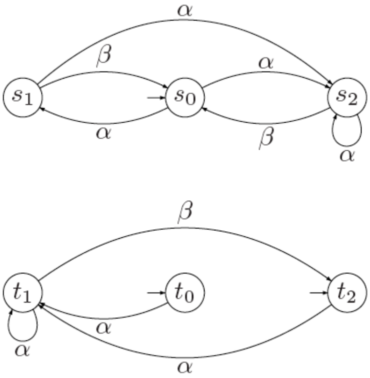
Consider the transition system $TS = TS_1 \uplus TS_2$ that results from the disjoint union of TS_1 and TS_2.

Then: TS_1 and TS_2 are called strongly bisimilar if there exists a strong bisimulation R on $S_1 \uplus S_2$ such that:

1. $\forall s \in I_{0,1}. \exists t \in I_{0,2}. (s, t) \in R$, and
2. $\forall t \in I_{0,2}. \exists s \in I_{0,1}. (s, t) \in R$.

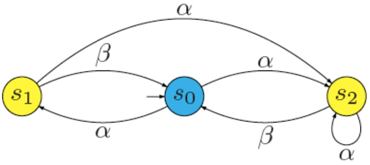
Example (1)

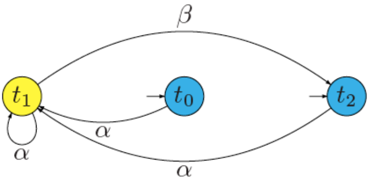
Are these transition systems strongly bisimilar? (No propositions.)



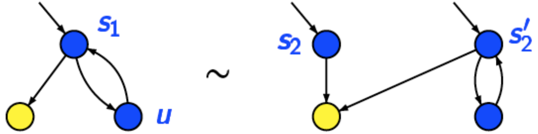
Example (2)

Yes, they are!



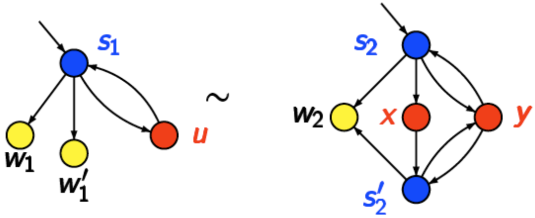


Correct or wrong?



wrong

$s_1 \longrightarrow u$, but $s_2 \not\longrightarrow$ *blue* (thus $s_1 \not\sim s_2$)



correct

bisimulation:

$\{(w_1, w_2), (w_1', w_2), (s_1, s_2), (s_1, s_2'), (u, x), (u, y)\}$

Quotient LTS under $\sim$

Quotient transition system

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Quotient transition system

For $TS = (S, Act, \longrightarrow, I_0, AP, L)$ and strong bisimilarity $\sim \subseteq S \times S$ let

$$TS/\sim \;=\; (S', Act, \longrightarrow', I'_0, AP, L'), \qquad \text{the } \textit{quotient} \text{ of } TS \text{ under } \sim$$

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Proof:

The binary relation:

$$R = \{ (s, [s]_\sim) \mid s \in S \}$$

is a strong bisimulation on the disjoint union $TS \uplus TS/\sim$.

Strong bisimulation revisited

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Auxiliary predicate

Let $P : S \times Act \times 2^S \to \{ 0, 1 \}$ be a predicate such that for $S' \subseteq S$:

$$P(s, \alpha, S') = \begin{cases} 1 & \text{if } \exists s' \in S'.\ s \xrightarrow{\alpha} s' \\ 0 & \text{otherwise.} \end{cases}$$

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Let $TS = (S, Act, \longrightarrow, I_0, AP, L)$ and R an *equivalence relation* on S.
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It can be easily proven that $\sim$ coincides with $\sim'$. Proof is omitted.

Overview

1. Strong Bisimulation
2. Probabilistic Bisimulation
   - Quotient Markov Chain
   - Examples
3. Logical Preservation
   - The Logics PCTL, PCTL* and PCTL$^-$
   - Preservation Theorem
4. Lumpability
5. Summary

Probabilistic bisimulation: intuition

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- Our aim: adapt bisimulation to discrete-time Markov chains.
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- When do two DTMC states exhibit the same step-wise behaviour?
- Key: if their transition probability for each equivalence class coincides.

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Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then: s is *probabilistic bisimilar* to t, denoted $s \sim_p t$, if there exists a probabilistic bisimulation R with $(s, t) \in R$.

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Remarks

As opposed to bisimulation on states in transition systems, any probabilistic bisimulation is an equivalence.

Example

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Bisimilar DTMCs

Let $\mathcal{D}_1$, $\mathcal{D}_2$ be DTMCs over the same set of atomic propositions with initial distributions ι^1_{init} and ι^2_{init}, respectively.

Consider the DTMC $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$ that results from the disjoint union of $\mathcal{D}_1$ and $\mathcal{D}_2$. Consider $\sim_p$ on $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$.

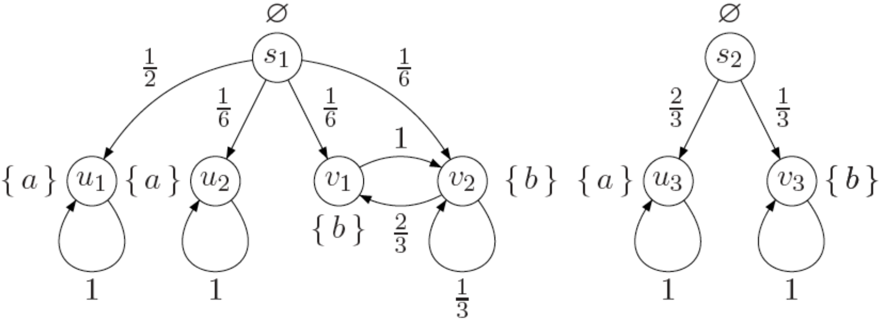
Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are bisimilar, denoted $\mathcal{D}_1 \sim_p \mathcal{D}_2$ whenever

$$\iota^1_{\mathrm{init}}(C) = \iota^2_{\mathrm{init}}(C)$$

for each bisimulation equivalence class C of $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$ under $\sim_p$.

Here, $\iota_{\mathrm{init}}(C)$ denotes $\sum_{s \in C} \iota_{\mathrm{init}}(s)$.

Example


∅
s_1
$\frac{1}{2}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
1
{ a } u_1
{ a } u_2
v_1
{ b }
$\frac{2}{3}$
v_2 { b }
1
1
$\frac{1}{3}$
∅
s_2
$\frac{2}{3}$
$\frac{1}{3}$
{ a } u_3
v_3 { b }
1
1

Quotient under $\sim_p$

Quotient DTMC under $\sim_p$

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For $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ and probabilistic bisimilarity $\sim_p \subseteq S \times S$ let

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The transition probability from $[s]_{\sim_p}$ to $[t]_{\sim_p}$ equals $\mathbf{P}(s, [t]_{\sim_p})$. This is well-defined as $\mathbf{P}(s, C) = \mathbf{P}(s', C)$ for all $s \sim_p s'$ and all bisimulation equivalence classes C.

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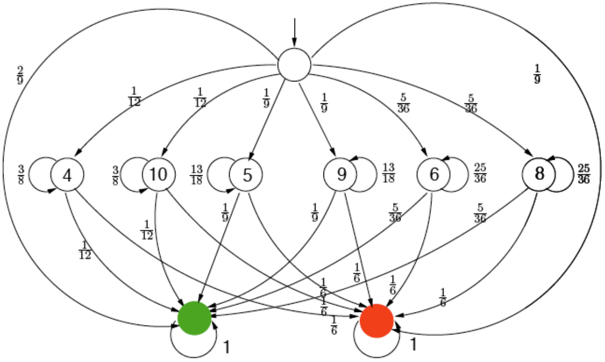
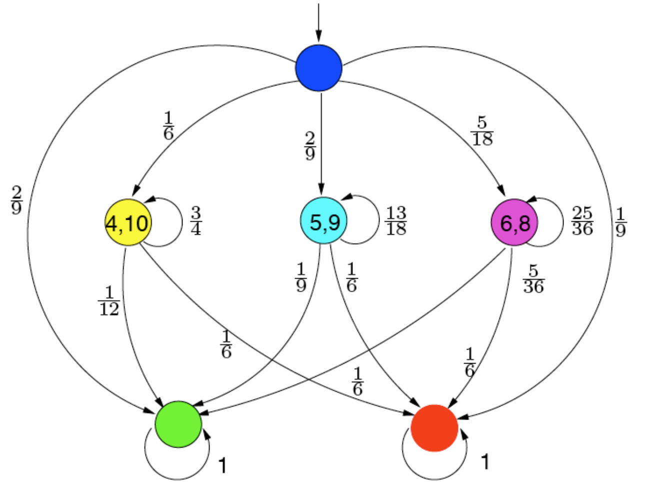


Quotient DTMC of Craps under $\sim_p$



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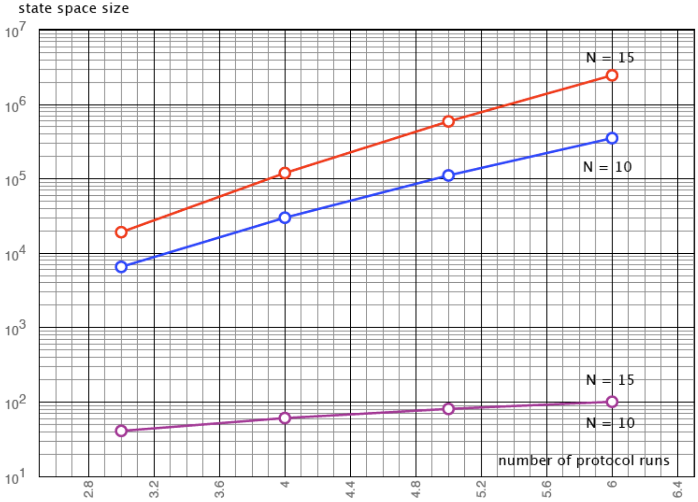
- Rebuild routing paths on crowd changes
- Property: Crowds protocol ensures "probable innocence":
  - probability real sender is discovered $< \frac{1}{2}$ if $N \geqslant \frac{p}{p-\frac{1}{2}} \cdot (c+1)$
  - where N is crowd's size and c is number of corrupt crowd members

State space reduction under $\sim_p$



IEEE 802.11 group communication protocol

	original DTMC			quotient DTMC		red. factor	
OD	states	transitions	ver. time	blocks	total time	states	time
4	1125	5369	122	71	13	**15.9**	**9.00**
12	37349	236313	7180	1821	642	**20.5**	**11.2**
20	231525	1590329	50133	10627	5431	**21.8**	**9.2**
28	804837	5750873	195086	35961	24716	**22.4**	**7.9**
36	2076773	15187833	5103900	91391	77694	**22.7**	**6.6**
40	3101445	22871849	7725041	135752	127489	**22.9**	**6.1**

all times in milliseconds

Overview

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Probabilistic Computation Tree Logic: Syntax

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A single PCTL-formula suffices!

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$\mathcal{D}, s \models \Phi$ if and only if state-formula Φ holds in state s of (possibly infinite) DTMC $\mathcal{D}$. As $\mathcal{D}$ is known from the context we simply write $s \models \Phi$.

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s \models \mathbb{P}_J(\varphi) & \text{iff} & Pr(s \models \varphi) \in J
\end{array}
$$

where $Pr(s \models \varphi) = Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\}$

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Measurability

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PCTL* measurability

For any PCTL* path formula φ and state s of DTMC $\mathcal{D}$,
the set $\{ \pi \in \mathit{Paths}(s) \mid \pi \models \varphi \}$ is measurable.

Proof:

Left as an exercise, using the result for PCTL measurability and the measurability of ω-regular properties.

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Examples in PCTL* but not in PCTL

$\mathbb{P}_{>\frac{1}{4}}(\bigcirc a \, \mathsf{U} \bigcirc b)$

Bounded until in PCTL*

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Bounded until can be defined using the other operators:

$$\varphi_1 \, \mathsf{U}^{\leqslant n} \varphi_2 \;=\; \bigvee_{0 \leqslant i \leqslant n} \psi_i \quad \text{where } \psi_0 = \varphi_2 \text{ and } \psi_{i+1} = \varphi_1 \wedge \bigcirc \psi_i \text{ for } i \geqslant 0.$$

Examples in PCTL* but not in PCTL

$\mathbb{P}_{>\frac{1}{4}}(\bigcirc a \, \mathsf{U} \bigcirc b)$ and $\mathbb{P}_{=1}(\mathbb{P}_{>\frac{1}{2}}(\Box\Diamond a \vee \Diamond\Box b))$.

Preservation of PCTL*-formulas

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Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

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2. By the last two results it follows that PCTL- and PCTL*-equivalence coincide. Thus any two states that satisfy the same PCTL formulas, satisfy the same PCTL* formulas.

PCTL$^-$ syntax

PCTL^{-} syntax

Simple Probabilistic Computation Tree Logic: Syntax

PCTL^{-} only consists of state-formulas. These formulas over the set AP obey the grammar:

$$\Phi \ ::= \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \Phi_1 \vee \Phi_2 \ \Big| \ \mathbb{P}_{\leqslant p}(\bigcirc \Phi)$$

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The next theorem shows that PCTL-, PCTL*- and PCTL$^-$-equivalence coincide.

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PCTL/PCTL* and Bisimulation Equivalence

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2. (b) $\Longrightarrow$ (c): trivial as PCTL is a sublogic of PCTL*.
3. (c) $\Longrightarrow$ (d): trivial as PCTL$-$ is a sublogic of PCTL.
4. (d) $\Longrightarrow$ (a): involved. First finite DTMCs, then for arbitrary DTMCs.

Proof

Overview

1960: Laurie Snell and John Kemeny

Lumpability

Ignore the initial distribution and state-labelling of a Markov chain.

Lumpability **[Kemeny & Snell, 1960]**

Let $\mathcal{D}$ be a (possibly countably infinite) DTMC with state space S and $\mathcal{B} = \{ B_1, \ldots, B_n \}$ be a partitioning of S (where B_j may be countably infinite). $\mathcal{D}$ is lumpable with respect to $\mathcal{B}$ iff for any B_i and B_j in $\mathcal{B}$ and any $s, s' \in B_i$:

$$\sum_{u \in B_j} \mathbf{P}(s, u) \; = \; \sum_{u \in B_j} \mathbf{P}(s', u) \quad \text{that is} \quad \mathbf{P}(s, B_j) = \mathbf{P}(s', B_j).$$

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It is easy to show that $S/\sim_p$ is a lumpable partition of the state space S. In fact, it is the coarsest possible lumpable partition.

Lumping equivalence

Lumping equivalence **[Kemeny & Snell, 1960]**

The DTMCs $\mathcal{D}$ and $\mathcal{D}'$ are lumping equivalent if there are lumpable partitions $\mathcal{B}$ of $\mathcal{D}$ and $\mathcal{B}'$ of $\mathcal{D}'$ such that there is an injective function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that:

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Corollary

$D \sim_p D'$ if and only if $\mathcal{D}$ and $\mathcal{D}'$ are lumping equivalent (with respect to the coarsest possible lumpable partition on their union).

Lumping equivalence

Remark

For finite Markov chains, the correspondence between lumping equivalence and $\sim_p$ allows to obtain the coarsest possible lumpable partition in an algorithmic, i.e., automated manner.

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This can be considered as a breakthrough in Markov chain theory.

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Take-home message

Probabilistic bisimulation coincides with a notion from the sixties, named (ordinary) lumpability.