Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1516/movep15/

Tuesday November 3, 2015

Overview

1. Reachability probabilities
2. What are qualitative properties?
3. Fairness theorem
4. Determining almost sure properties
   - Preliminaries
   - Long run theorem
   - Reachability, repeated reachability and persistence
   - Quantitative repeated reachability and persistence
5. Summary

Recapitulating reachability probabilities

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5. Intermediate results $\mathbf{x}^{(i)}$ represent the vector $\big(Pr(s \models \Diamond^{\leqslant i} G) \big)_{s \in S_?}$

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Comparing the probability of an event such as $\Box G$, $\Diamond\Box G$ and $\Box\Diamond G$ with a threshold $\sim p$ with $p \in (0, 1)$ and $\sim$ a binary comparison operator $(=, <, \leqslant, \geqslant, >)$ yields a quantitative property.

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In the following we will concentrate on almost sure events, i.e., events E with $Pr(E) = 1$.

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Take-home message

For finite DTMCs, qualitative properties do only depend on their state graph and not on the transition probabilities! For infinite DTMCs, this does not hold.

Remark

In the following we will concentrate on almost sure events, i.e., events E with $Pr(E) = 1$. This suffices, as $Pr(E) > 0$ if and only if not $Pr(\overline{E}) = 1$.

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Corollary

For any state s in a (possibly infinite) DTMC we have:

$$Pr(s \models \bigwedge_{t \in S} \bigwedge_{u \in Post^*(t)} (\Box\Diamond t \;\Rightarrow\; \Box\Diamond u)) \;=\; 1.$$

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This result follows directly from the following claim that we will prove below.

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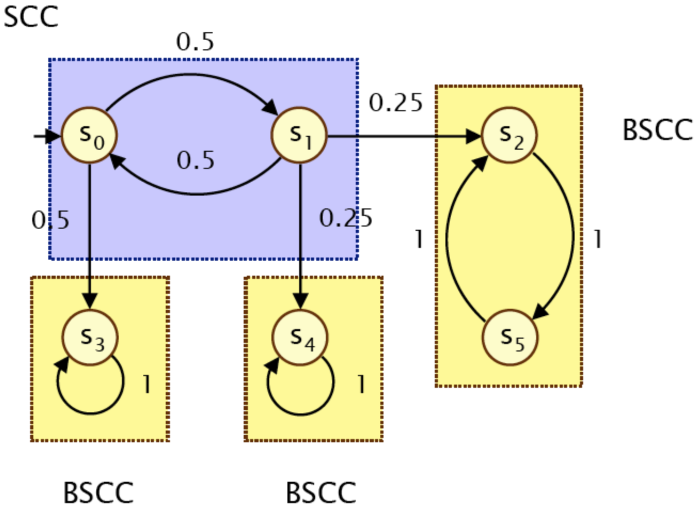
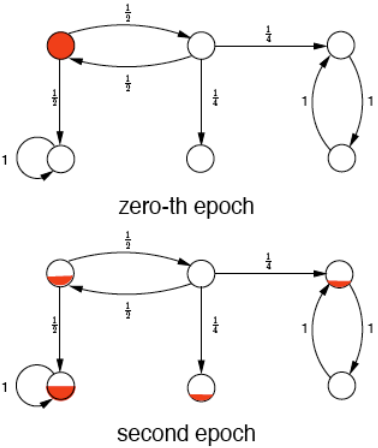
- Let $BSCC(\mathcal{D})$ denote the set of BSCCs of DTMC $\mathcal{D}$.

Example



Evolution of an example DTMC



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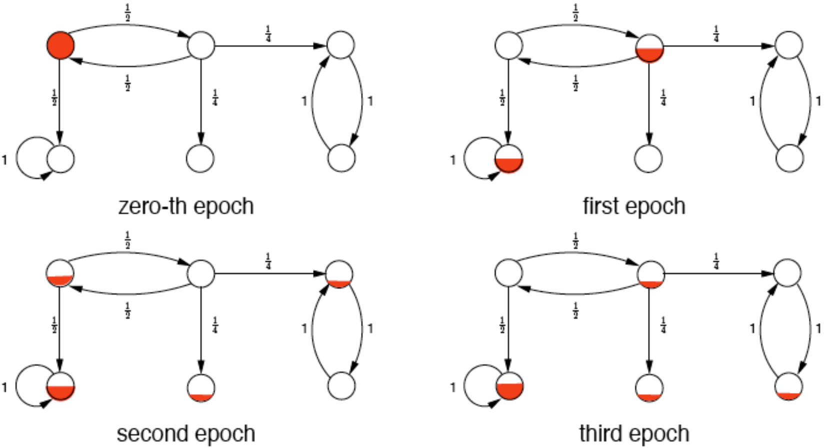
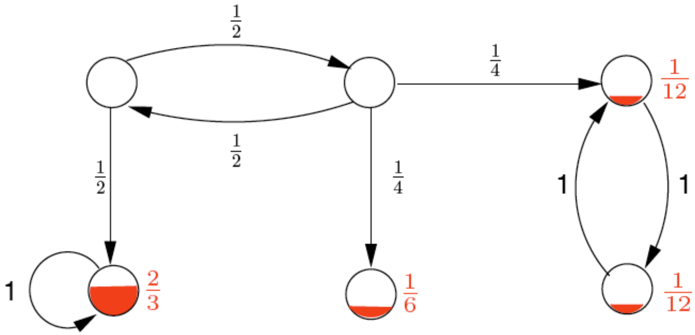


Which states have a probability > 0 when repeating this on the long run?

On the long run



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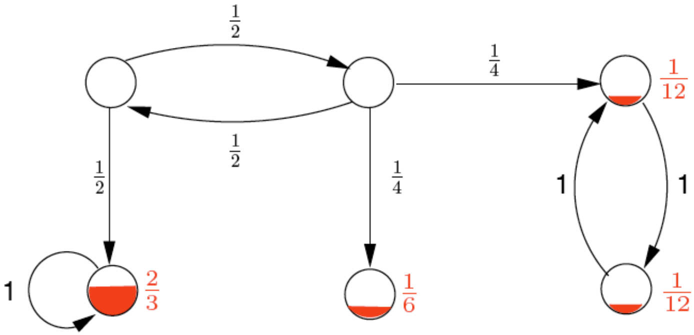

The probability mass on the long run is only left in BSCCs.

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Intuition

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- As $\mathcal{D}$ is finite, $\inf(\pi)$ is strongly connected, i.e., part of SCC T, say.

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$$Post^*(T) = Post^*(\inf(\pi)) \subseteq \inf(\pi) = T.$$

- Hence, $T = Post^*(T)$, i.e., T is a BSCC. The claim follows from $(*)$.

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Let p be probability that no reply is received on a probe.

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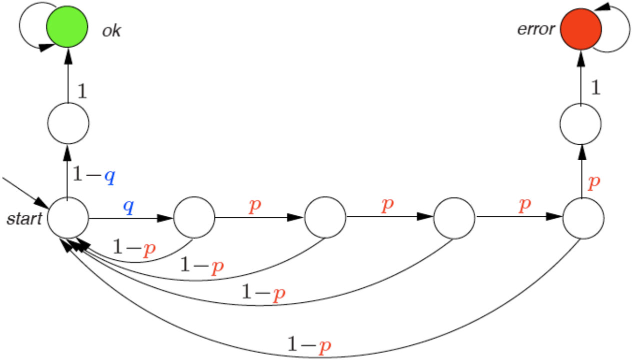

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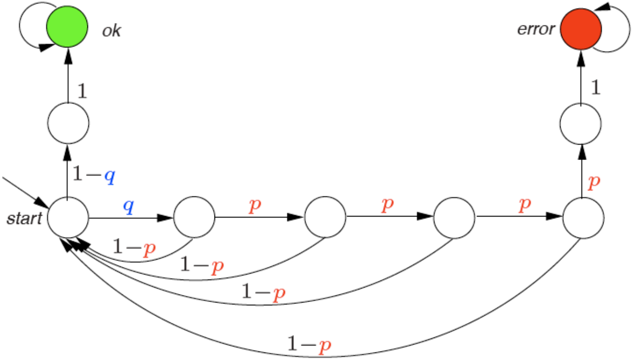

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By the long-run theorem, the probability of acquiring an address infinitely often is zero.

Almost sure reachability

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Recall: an absorbing state in a DTMC is a state with a self-loop with probability one.

Almost sure reachability theorem

For finite DTMC with state space S, $s \in S$ and $G \subseteq S$ a set of absorbing states:

$$Pr(s \models \Diamond G) = 1 \quad \text{iff} \quad s \in S \setminus Pre^*\big(\, S \setminus Pre^*(G)\,\big).$$

Note: $S \setminus Pre^*\big(\, S \setminus Pre^*(G)\,\big)$ are states that cannot reach states from which G cannot be reached.

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Show that both sides of the equivalence are equivalent to $Post^*(t) \cap G \neq \varnothing$ for each state $t \in Post^*(s)$. Rather straightforward.

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This yields a time complexity which is linear in the size of the DTMC $\mathcal{D}$.

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Proof:

Immediate consequence of the long-run theorem.

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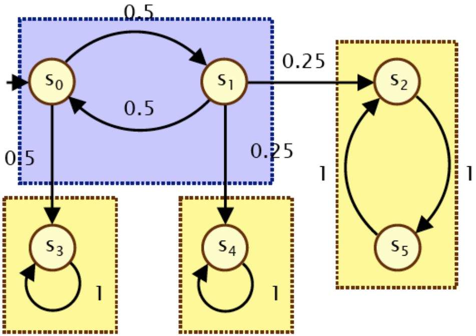

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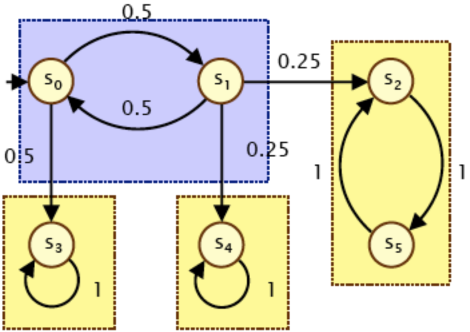

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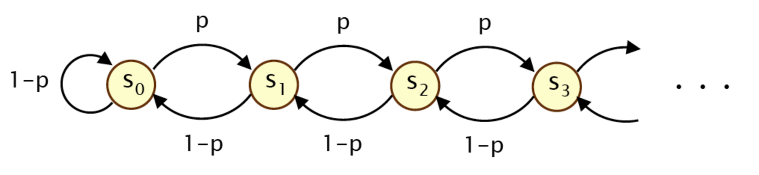

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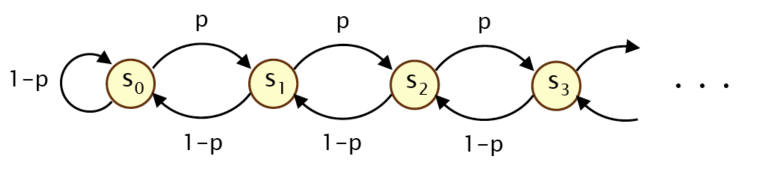

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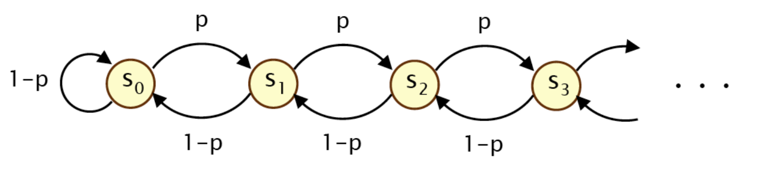

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Remark

Thus probabilities for $\Box\Diamond G$ and $\Box\Diamond G$ are reduced to reachability probabilities. These can be computed by solving a linear equation system.

Example

Overview

1 Reachability probabilities

2 What are qualitative properties?

3 Fairness theorem

4 Determining almost sure properties
- Preliminaries
- Long run theorem
- Reachability, repeated reachability and persistence
- Quantitative repeated reachability and persistence

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