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Exercise 1 (Sigma Algebras): (4 points)

Let Ω be a countably infinite set and define $\mathfrak{F}_\Omega$ as the smallest class of subsets of Ω such that for all $A \subseteq \Omega$

(i) if A is finite, then $A \in \mathfrak{F}_\Omega$, and

(ii) if $A \in \mathfrak{F}_\Omega$, then $A^c \in \mathfrak{F}_\Omega$ for $A^c := (\Omega \setminus A)$.

a) Show that the definition is non-trivial, i.e., in general $\mathfrak{F}_\Omega \neq 2^\Omega$.
(Hint: find a set Ω and a subset $A \subseteq \Omega$ which cannot be in $\mathfrak{F}_\Omega$ according to the above definition.)

b) Would this change if $\mathfrak{F}_\Omega$ is defined as the *largest* class of subsets defined as above (instead of the *smallest*)?

c) Prove or disprove that $\mathfrak{F}_\Omega$ is a σ-algebra as defined in the lecture for any countably infinite set Ω.

Exercise 2 (Geometric Distribution): (3 + 3 points)

Recall the definition of a *geometric distribution* as given in the lecture:

Definition 1. *Let X be a discrete random variable, $k \in \mathbb{N}_{>0}$ and $0 < p \leq 1$. The mass function of a* geometric distribution *is given by:*

$$\Pr\{X = k\} = (1-p)^{k-1} \cdot p$$

Let X now be be geometrically distributed with parameter p.

a) Show that $E[X] = \frac{1}{p}$ and $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

b) Prove that

$$\Pr\{X = k + m \mid X > m\} \;=\; \Pr\{X = k\} \quad \text{for any} \quad m, k \in \mathbb{N}_{>0}.$$

Hint: Use properties of probability measures and the geometric distribution as presented in the lecture.