Concurrency Theory

True Concurrency Semantics of Petri Nets (I)

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http://moves.rwth-aachen.de/teaching/ws-1516/ct

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Overview

1. Introduction
2. Nets and markings
3. The true concurrency semantics of Petri nets
4. Distributed runs
5. Summary

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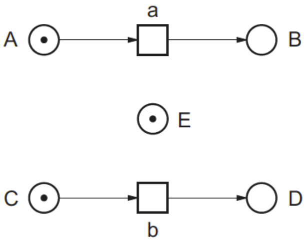
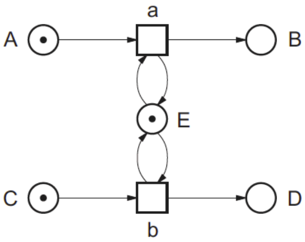
Motivation





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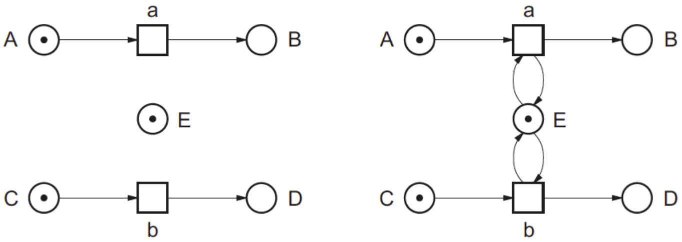

Nets with identical sequential runs (*a* occurs before *b*, or vice versa), but the left net allows the simultaneous execution of *a* and *b* whereas the right one does not.

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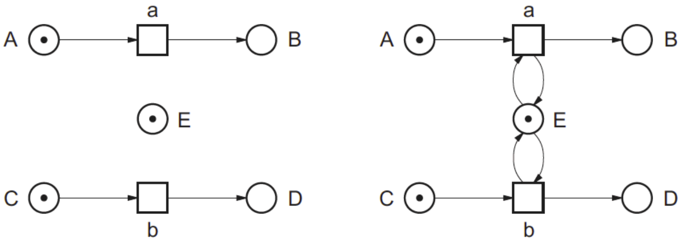

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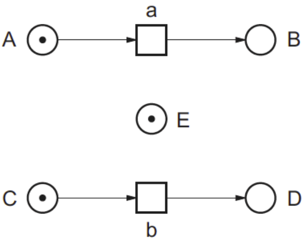

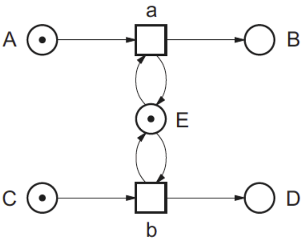

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This requires a finer perspective on transition execution.

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Nets

Net

A Petri net N is a triple (P, T, F) where:

- P is the countable set of places
- T is the countable set of transitions with $P \cap T = \varnothing$
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs.

Places and transitions are generically called nodes.
We assume that ${}^{\bullet}t$ and $t^{\bullet}$ are finite, for each $t \in T$.

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Marking

A marking M of a net $N = (P, T, F)$ is a mapping $M : P \to \mathbb{N}$.
For net $N = (P, T, F)$ and marking M_0, the tuple (P, T, F, M_0) is called an elementary system net. M_0 is the initial marking of N.

Transition occurrence

Enabling and occurrence of a transition

A marking M enables a transition t if $M(p) \geqslant 1$ for each place $p \in {}^\bullet t$.

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Transition t can occur in marking M if t is enabled at M. Its occurrence leads to marking M', denoted $M \xrightarrow{t} M'$, defined for place $p \in P$ by:

$$M'(p) \;=\; M(p) - F(p,t) + F(t,p).$$

where we represent F by its characteristic function.

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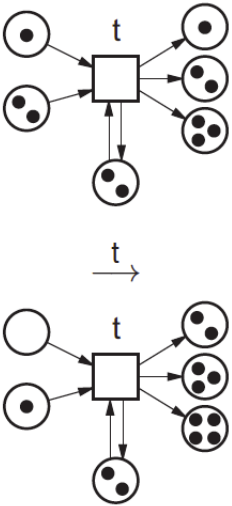

Reachable markings

Step sequence

A sequence of transitions $\sigma = t_1\, t_2 \ldots t_n$ is a step sequence if there exist markings M_1 through M_n such that:

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M_n$$

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M is a reachable marking if there exists a step sequence σ with $M_0 \xrightarrow{\sigma} M$.

Sequential runs

Sequential run

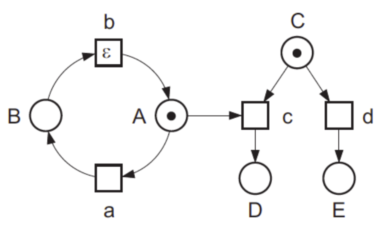
Let N be an elementary net system. A sequential run of N is a sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots$$

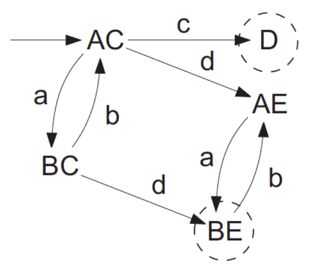
of steps of N starting with the initial marking M_0. A run can be finite or infinite. A finite run $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_1} \cdot \xrightarrow{t_n} M_n$ is complete if M_n does not enable any transition.

Marking graph

The marking graph of N has as nodes the reachable markings of N and as edges the reachable steps of N.



A sample elementary net system



Its marking graph

The interleaving semantics of Petri nets

The interleaving semantics of a Petri net is its marking graph.

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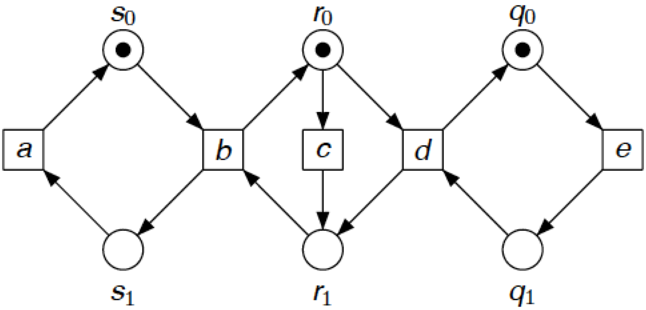

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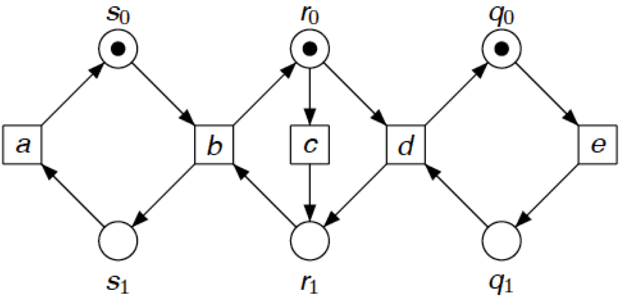

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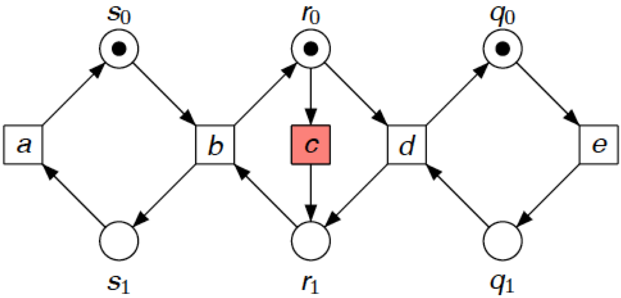

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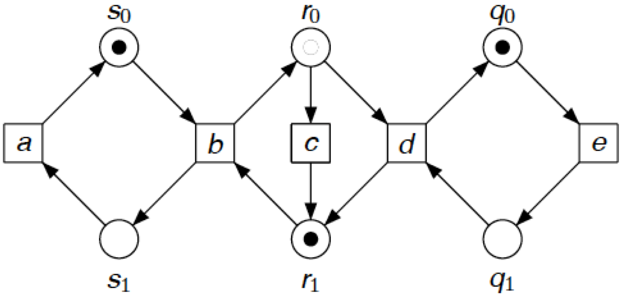

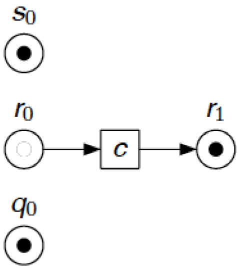

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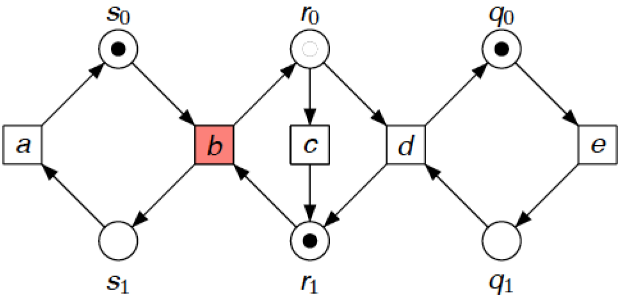

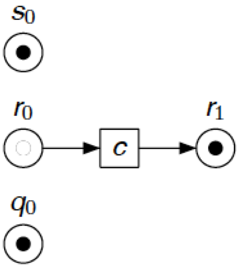

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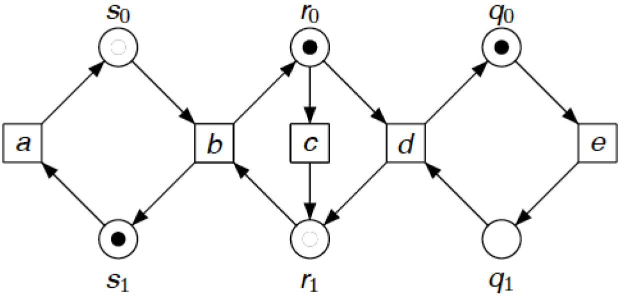

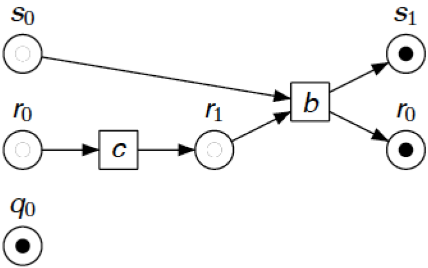

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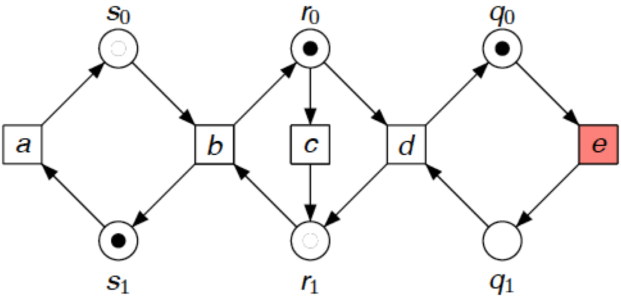

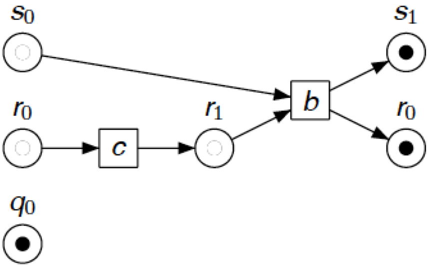

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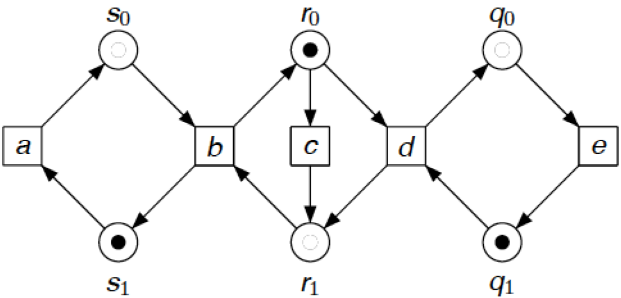

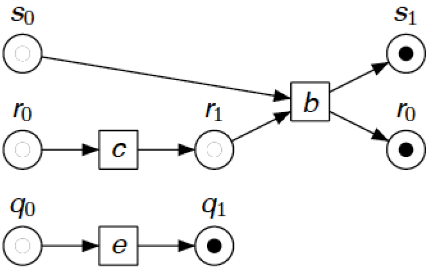

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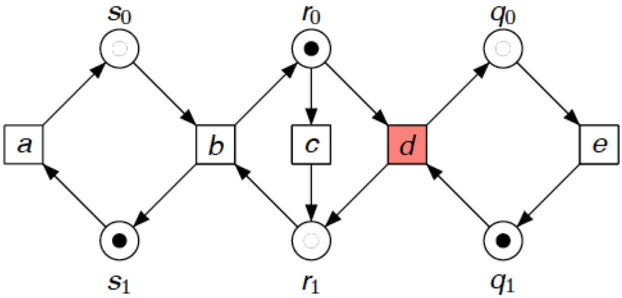

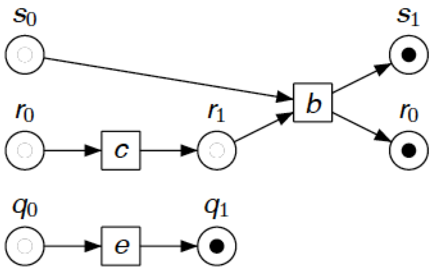

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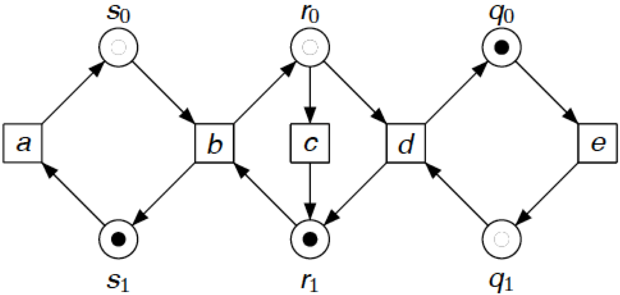

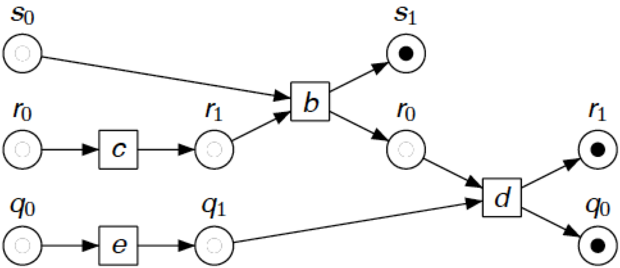

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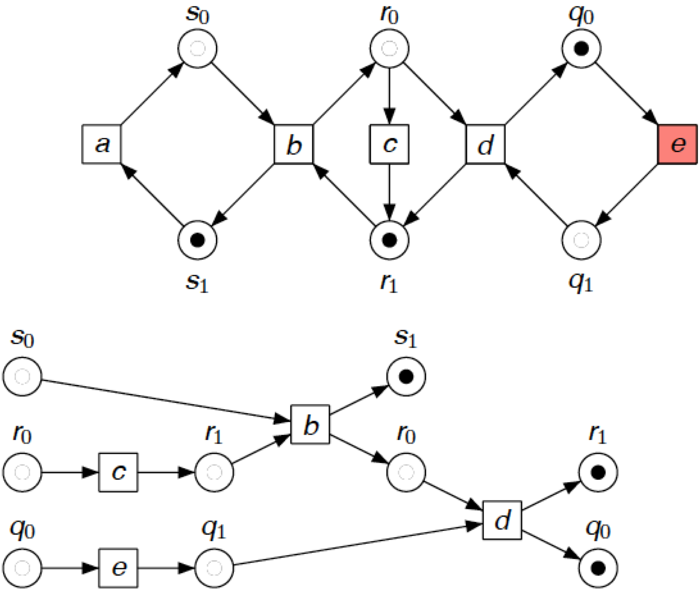

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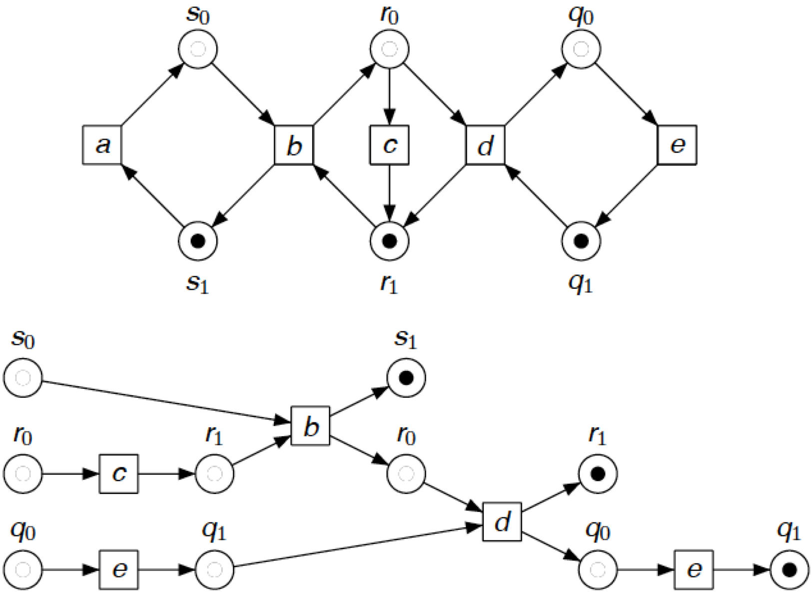

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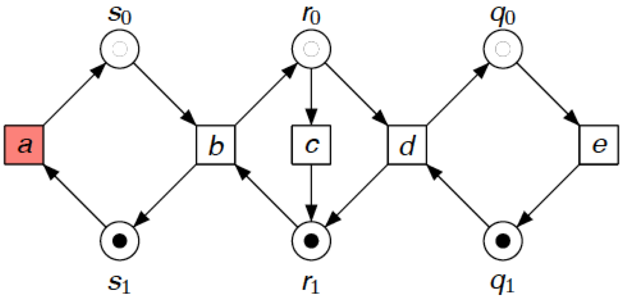

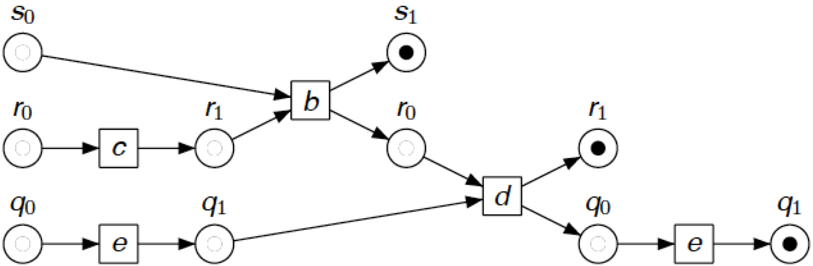

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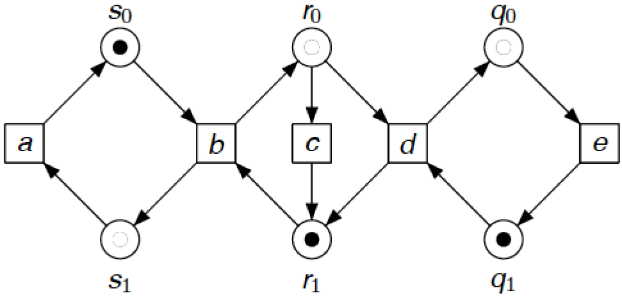

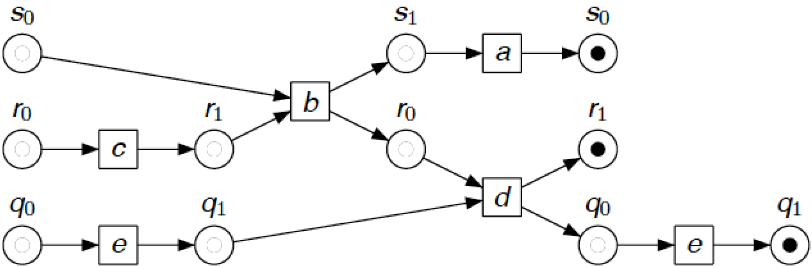

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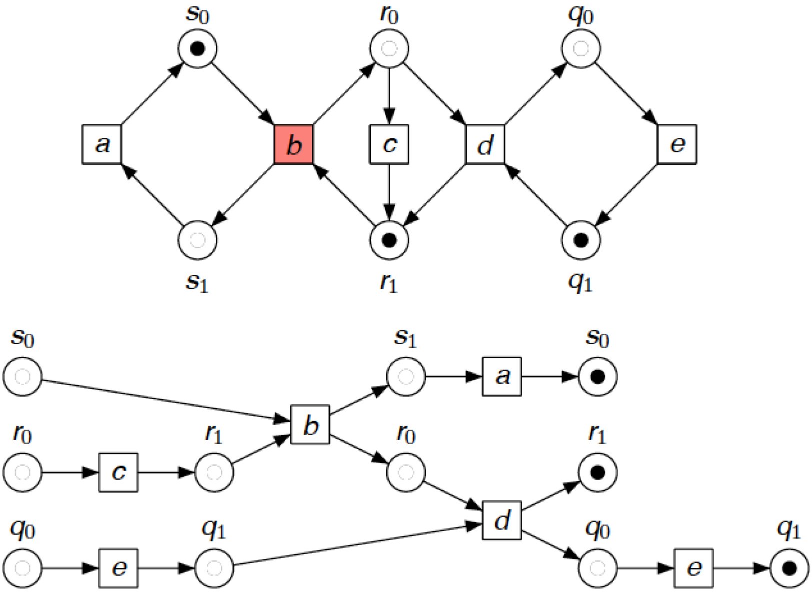

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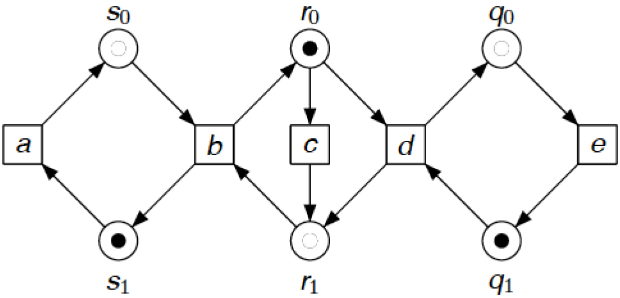

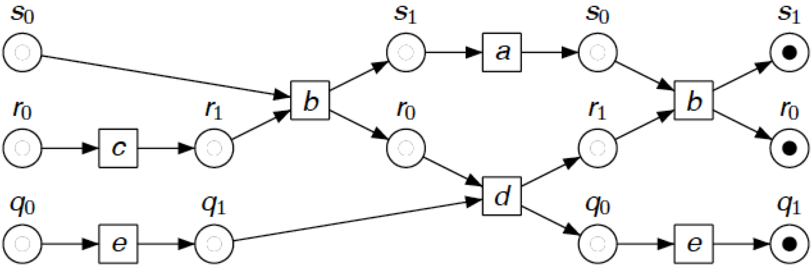

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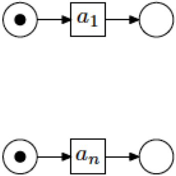
The true concurrency thesis:

> *The total order assumption does not correspond to physical reality and leads to awkward representations of simple phenomena*

Interleaving versus true concurrency

In interleaving semantics, a system composed of n independent components



has $n!$ different executions

The automaton accepting them has 2^n states

In true concurrency semantics, it has only one nonsequential execution

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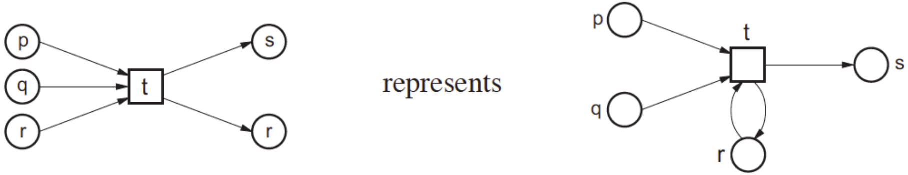

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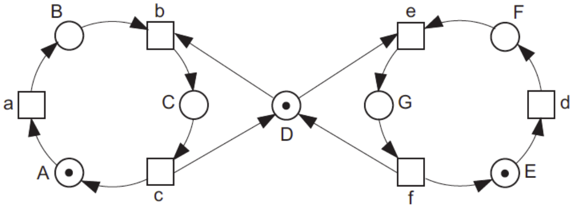
Mutual exclusion net and its actions



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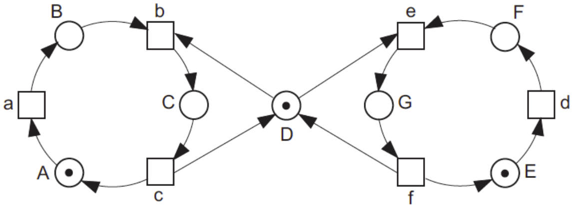
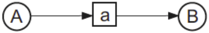
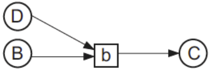
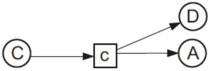
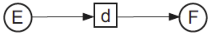
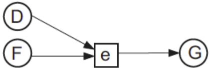


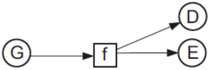
N_a :


N_b :


N_c :


N_d :


N_e :


N_f :


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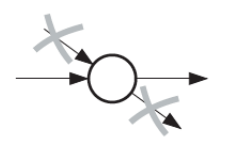
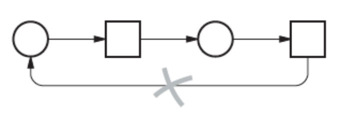
Intuition

No place branches, no sequence of arcs forms a loop, and each sequence of arcs has a first node.





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Note: the "runs" of the example net (with initial marking) are all causal nets

Properties of causal nets

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Let $N = (P, T, F, M_0)$ be a causal net. Then every step sequence:

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By contraposition.

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Proof.

Follows directly from the fact that the initial marking M_0 is one-bounded, and by the above lemma. □

Completeness of a causal net

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Absence of superfluous places and transitions

Let $N = (P, T, F, M_0)$ be a causal net. Then there exists a possibly infinite step sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots\cdots \xrightarrow{t_k} M_k \xrightarrow{t_{k+1}} \cdots$$

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On the black board. □

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On the black board. □

A causal net thus contains no superfluous places and transitions, as every place is visited and every transition is fired in the above step sequence.

Outset and end of a causal net

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The outset and end of causal net $K = (Q, V, G, M)$ are defined by:

$$^\circ K = \{ q \in Q \mid {}^\bullet q = \varnothing \} \quad \text{and} \quad K^\circ = \{ q \in Q \mid q^\bullet = \varnothing \}.$$

Places without an incoming arc form the outset $^\circ K$. The places without an outgoing arc form the end K°.

What is a distributed run?

Distributed run

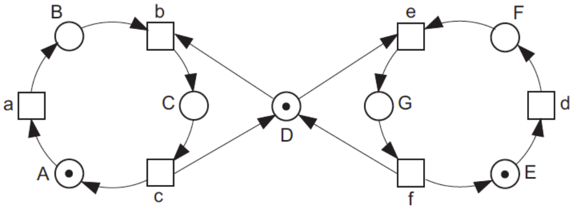
A distributed run of a one-bounded elementary net system N is:

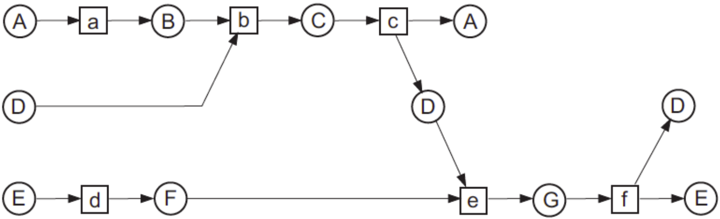
1. a labeled causal net K_N
2. in which each transition t (with ${}^\bullet t$ and $t^\bullet$) is an action of N.

A distributed run K_N of N is complete iff (the marking) ${}^\circ K$ represents the initial marking of N and (the marking) K_N° does not enable any transition.

If N is clear from the context we just write K for K_N.

A distributed run for mutual exclusion





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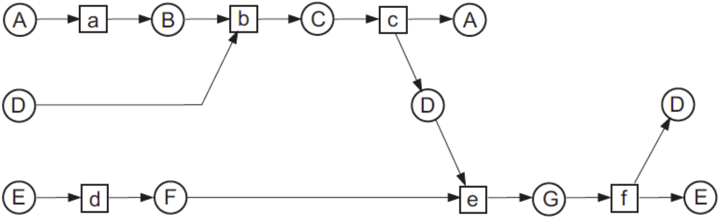


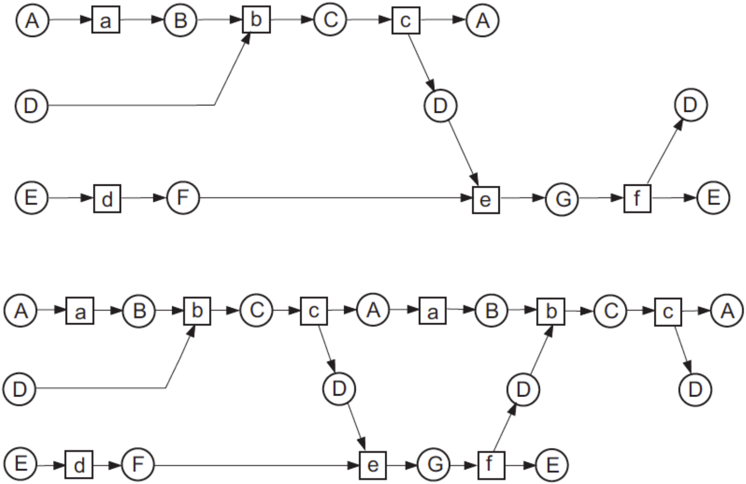
Distributed run of the mutual exclusion algorithm.

Actions N_a, N_b, N_c and N_d causally precede N_e. They form a chain.

N_a and N_d are not linked by actions; they are causally independent.

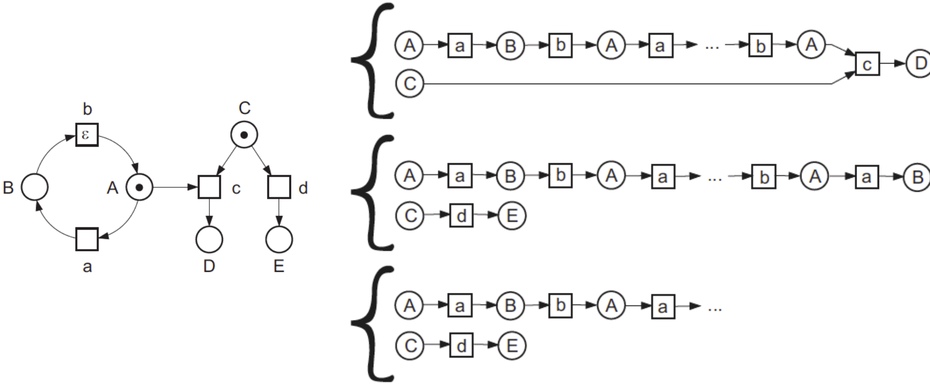
The same applies to N_b and N_d and N_c and N_d.

Expansion of a distributed run for mutual exclusion



A distributed run (top) and its extension with actions *b* and *c*.

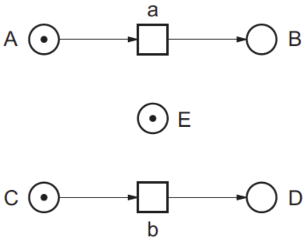
More distributed runs

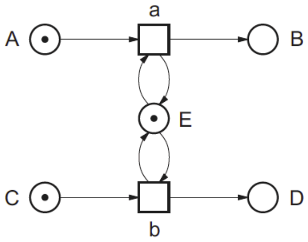


Various finite distributed runs and an infinite distributed run (right) of net (left).

Causal order

Opposed to sequential runs, distributed runs show the causal order of actions.





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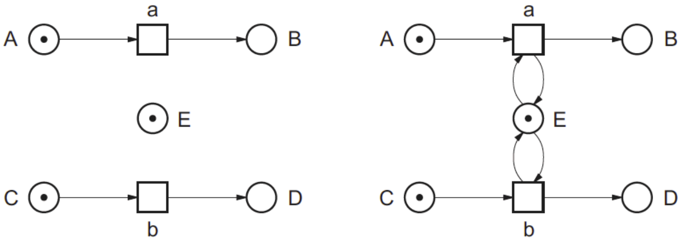
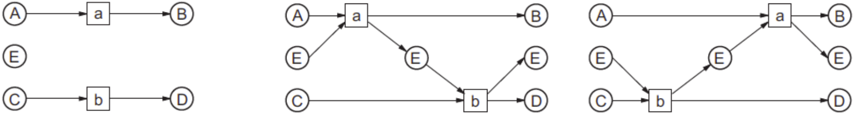


Nets with identical sequential runs (a occurs before b, or vice versa), but the left net has the left distributed run below, the right net both other ones:



Composition of distributed runs

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For $i = 1, 2$, let $K_i = (Q_i, V_i, G_i)$ be causal nets, labeled with ℓ_i. Let $(Q_1 \cup V_1) \cap (Q_2 \cap V_2) = K_1^\circ = {}^\circ K_2$ and for each place $p \in K_1^\circ$ let $\ell_1(p) = \ell_2(p)$.

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Intuition

The composition $K \bullet L$ is formed by identifying the end K° of K with the outset ${}^\circ L$ of L. To do this, K° and ${}^\circ L$ must represent the same marking.

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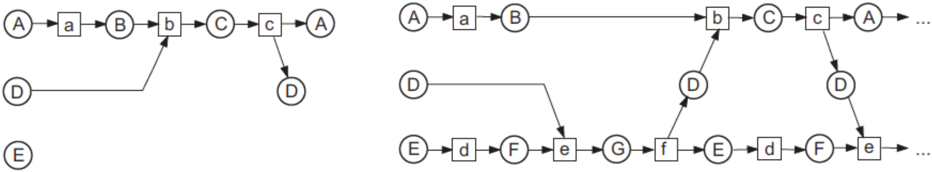

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Today: a characterization of distributed runs using homomorphisms.

Net homomorphisms

[2]Here $h(X)$ for set X of nodes is defined by $h(X) = \bigcup_{x \in X} h(x)$.

[3]Due to the 1-boundedness, a marking M is a subset of the set P of places.

Net homomorphisms

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A homomorphism from $N_1 = (P_1, T_1, F_1, M_{0,1})$ to $N_2 = (P_2, T_2, F_2, M_{0,2})$ is a mapping $h : P_1 \cup T_1 \to P_2 \cup T_2$ such that:

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1. $h(P_1) \subseteq P_2$ and $h(T_1) \subseteq T_2$, and
2. $\forall t \in T_1$, the restriction of h to ${}^\bullet t$ is a bijection between ${}^\bullet t$ (in N_1) and ${}^\bullet h(t)$ (in N_2), and similarly for $t^\bullet$ and $h(t)^\bullet$, and

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Net homomorphisms

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A homomorphism from $N_1 = (P_1, T_1, F_1, M_{0,1})$ to $N_2 = (P_2, T_2, F_2, M_{0,2})$ is a mapping $h : P_1 \cup T_1 \rightarrow P_2 \cup T_2$ such that: [2]

1. $h(P_1) \subseteq P_2$ and $h(T_1) \subseteq T_2$, and
2. $\forall t \in T_1$, the restriction of h to $^\bullet t$ is a bijection between $^\bullet t$ (in N_1) and $^\bullet h(t)$ (in N_2), and similarly for $t^\bullet$ and $h(t)^\bullet$, and
3. the restriction of h to $M_{0,1}$ is a bijection between $M_{0,1}$ and $M_{0,2}$.[3]

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Intuition

A homomorphism is a mapping between nets that preserves the nature of nodes and the environment of nodes. A homomorphism from N_1 to N_2 means that N_1 can be folded onto a part of N_2, or in other words, that N_1 can be obtained by partially unfolding a part of N_2.

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Distributed run using homomorphisms

Distributed run **[Best and Fernandez, 1988]**

A distributed run of an elementary net system N is a pair (K, h) where K is a causal net and h is a homomorphism from K to N.[4]

[4] Best and Fernandez called this a process of a net.

[5] In the previous lecture, the labeling h was explicitly given as ℓ.

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A distributed run (K, h) of N may be viewed as a net K of which the places and transitions are labeled by places and transitions of N such that the labeling h forms a net homomorphism from K to N.[5]

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Examples

Overview

1. Introduction
2. Nets and markings
3. The true concurrency semantics of Petri nets
4. Distributed runs
5. Summary

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- A causal net is a possibly infinite net which is:
  - well-founded, acyclic, and has no place branching, and
  - whose initial marking are the places without incoming arcs

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- Nets that have the same causal nets are causally equivalent
- Distributed run = the "true concurrency" analogue to a sequential run