Concurrency Theory

Interleaving Semantics of Petri Nets

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Overview

Overview

Carl Adam Petri (1926-2010)

The original work[1] does not contain a single (graphical) Petri net!

[1] Petri's PhD dissertation, 1962.

Semantics: executions and traces

Models in the 60s: lambda calculus, finite automata, Turing machines, ...

States: current configurations of the machine

- One or more initial states
- Possibly some distinguished final states

Transitions: moves between configurations

Lambda calculus	$(\lambda x.xx)(\lambda y.y)$	$\longrightarrow$	$(\lambda y.y)(\lambda z.z)$
Turing machine	$0010q_1011$	$\longrightarrow$	$001q_201011$
Finite automaton	q_1	$\xrightarrow{a}$	q_2
Pushdown automaton	$(q_1, XYYZ)$	$\xrightarrow{a}$	$(q_2, XYXYYZ)$

Executions: alternating sequences of states and transitions

Petri's question

C.A. Petri points out a discrepancy between how Theoretical Physics and Theoretical Computer Science described systems in 1962:

Theoretical Physics describes systems as a collection of interacting particles (subsystems), without a notion of global clock or simultaneity

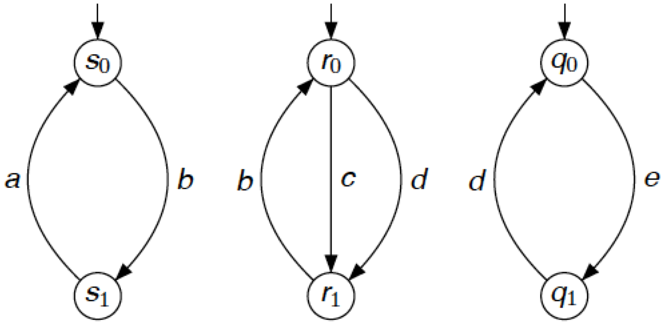
Theoretical Computer Science describes systems as sequential virtual machines going through a temporally ordered sequence of global states

Petri's question:

Which kind of abstract machine should be used to describe the physical implementation of a Turing machine?

Petri net

A graphical representation of interacting finite automata:



Petri net

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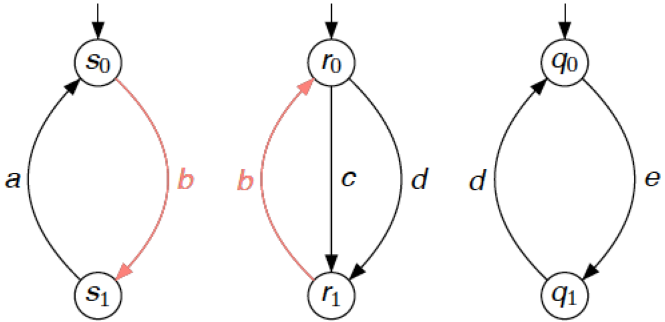

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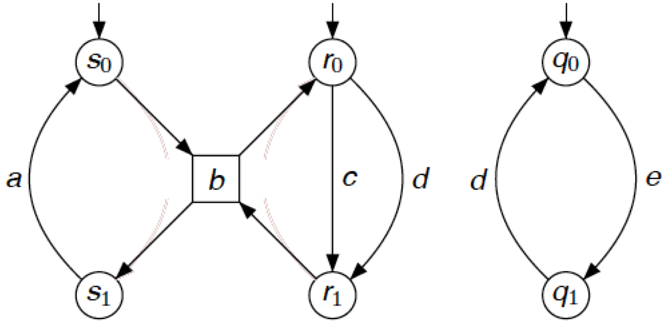

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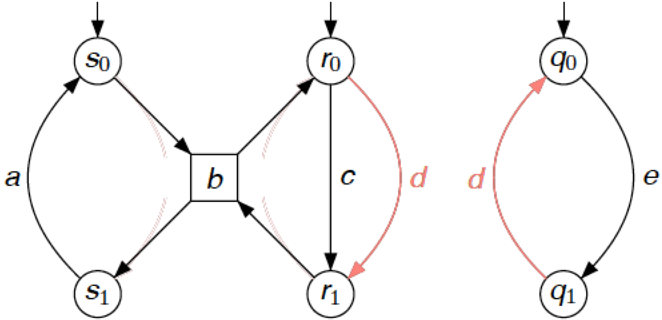

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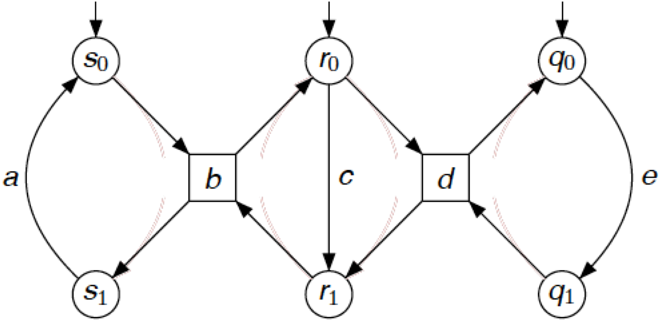

Petri net

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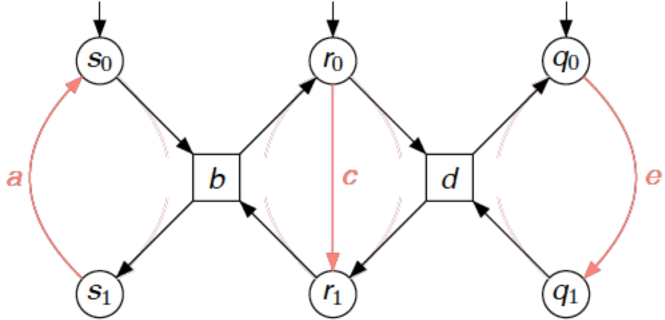

Petri net

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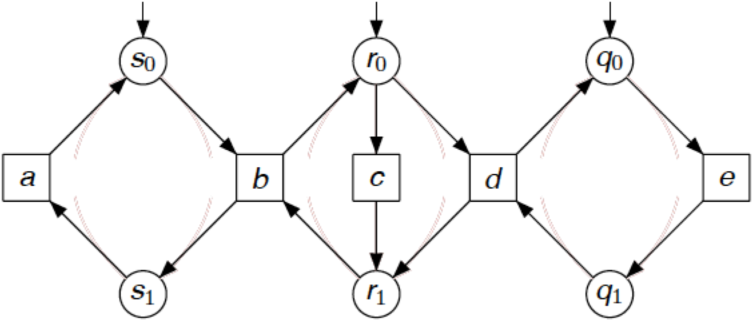

Petri net

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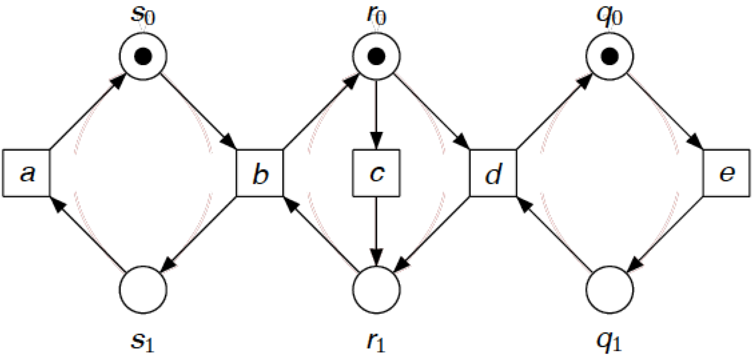


Overview

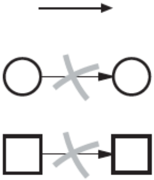
Components of a net

A Petri net is a structure with two kinds of elements: places and transitions. They are connected by arcs.

A place is represented by a circle or ellipse. A place p always models a passive component: p can store, accumulate or show things.

A transition is represented by a square or rectangle. A transition t always models an active component: t can produce things, consume, transport or change them.

Places and transitions are connected to each other by directed arcs. Graphically, an arc is represented by an arrow. An arc models an abstract, sometimes only notional relation between components. Arcs run from places to transitions or vice versa.



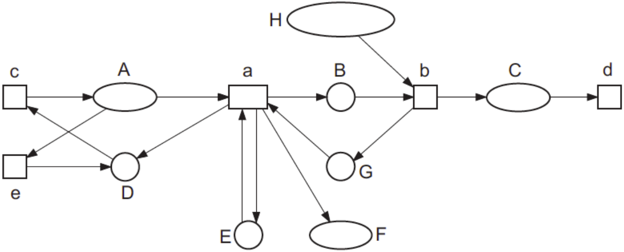
Nets

Net

A Petri net N is a triple (P, T, F) where:

- P is the finite set of places
- T is the finite set of transitions with $P \cap T = \varnothing$
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs[2]

Places and transitions are generically called nodes.



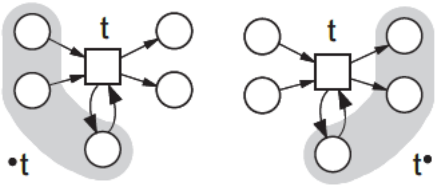
[2] F is also called the flow relation.

The pre- and post-sets

Pre- and post-sets

Let node $x \in P \cup T$.
The pre-set of x is defined by: ${}^{\bullet}x = \{ y \mid (y,x) \in F \}$.
The post-set of x is defined by: $x^{\bullet} = \{ y \mid (x,y) \in F \}$.

Two nodes $x, y \in N$ form a loop if $x \in {}^{\bullet}y$ and $y \in {}^{\bullet}x$.

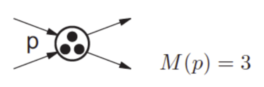


Markings

Marking

A marking M of a net $N = (P, T, F)$ is a mapping $M : P \to \mathbb{N}$.

For net $N = (P, T, F)$ and marking M_0, the tuple (P, T, F, M_0) is called an elementary net. M_0 is the initial marking of N.



Intuition

Note: a marking is a multiset. It defines a distribution of tokens across places. Tokens are depicted as black dots.

Transition firing

Enabling and firing of a transition

Let (P, T, F, M) be an elementary net. Marking M enables a transition t if $M(p) \geqslant 1$ for each place $p \in {}^\bullet t$.

Transition t can fire in marking M if t is enabled at M. Its firing leads to marking M', denoted by step $M \xrightarrow{t} M'$, defined for place $p \in P$ by:

$$M'(p) \;=\; M(p) - F(p,t) + F(t,p).$$

where we represent F by its characteristic function.

Intuition

Transition t is enabled whenever every $p \in {}^\bullet t$ holds at least one token. On t's firing, one token is removed from each place in ${}^\bullet t$, and one token is put in each place in $t^\bullet$:

$$M'(p) \;=\; \begin{cases} M(p) - 1 & \text{if } p \in {}^\bullet t \text{ and } p \notin t^\bullet \\ M(p) + 1 & \text{if } p \in t^\bullet \text{ and } p \notin {}^\bullet t \\ M(p) & \text{otherwise} \end{cases}$$

Transition firing

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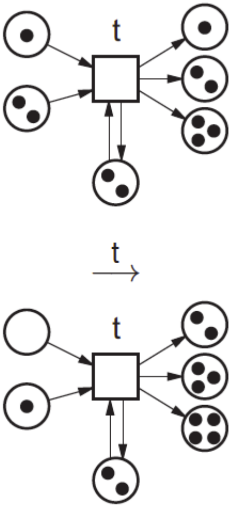


Overview

The interleaving semantics of Petri nets

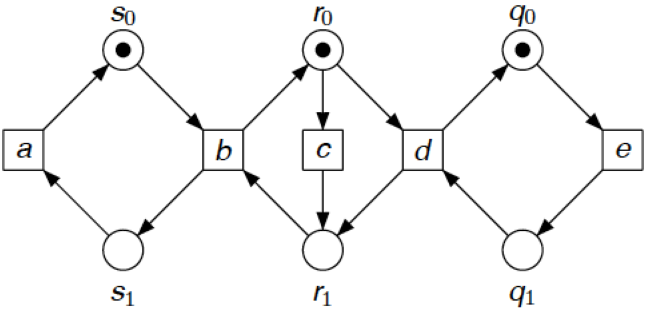
An execution semantics

States: markings (distributions of tokens over the net)

Transitions: $M \xrightarrow{t} M'$

Sequential runs: $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \dots$

The interleaving semantics of Petri nets

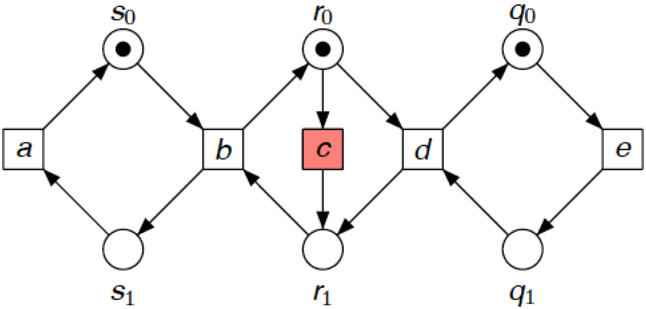


$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As the marking for s_0 is the complement of s_1, the marking for s_0 is omitted.

The same applies to the places r_0 and q_0.

The interleaving semantics of Petri nets



$$\begin{matrix} s_1 \\ r_1 \\ q_1 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c}$$

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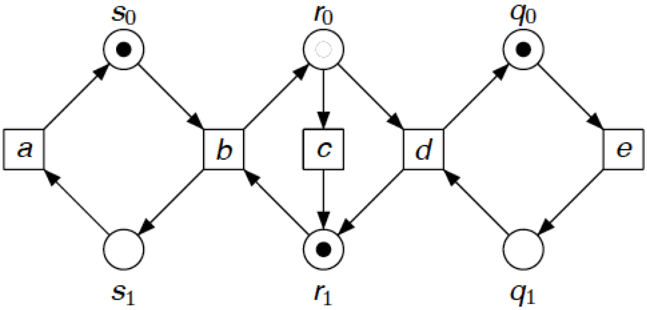

$$\begin{matrix} s_1 \\ r_1 \\ q_1 \end{matrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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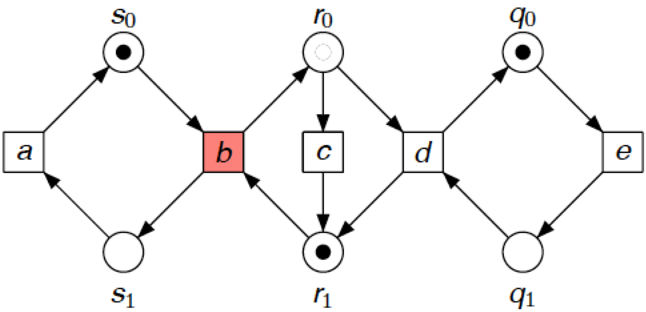

$$\begin{matrix} s_1 \\ r_1 \\ q_1 \end{matrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b}$$

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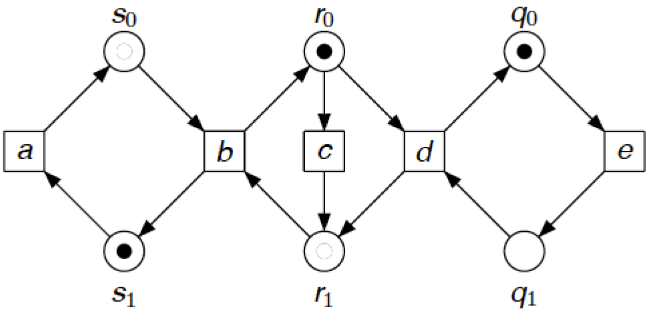

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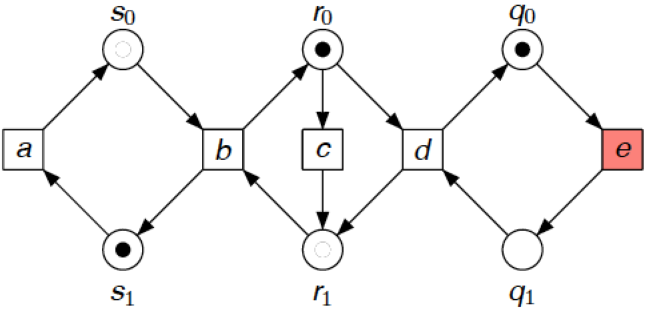

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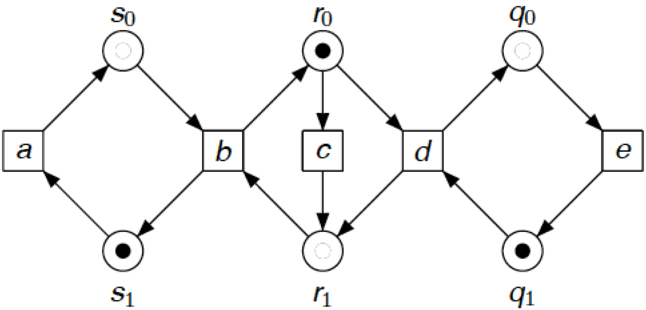

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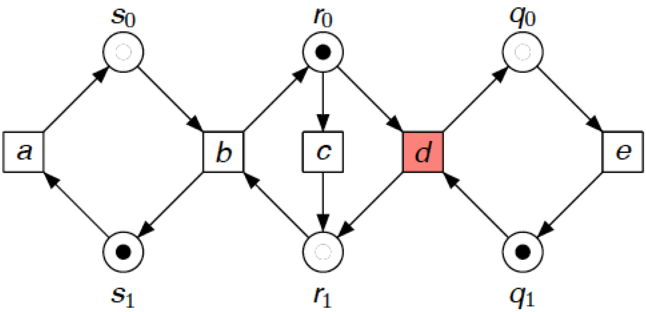

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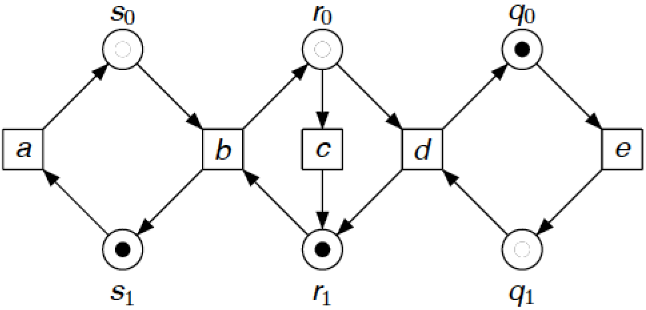

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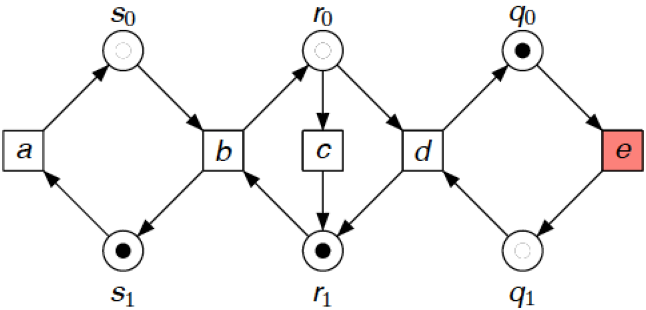

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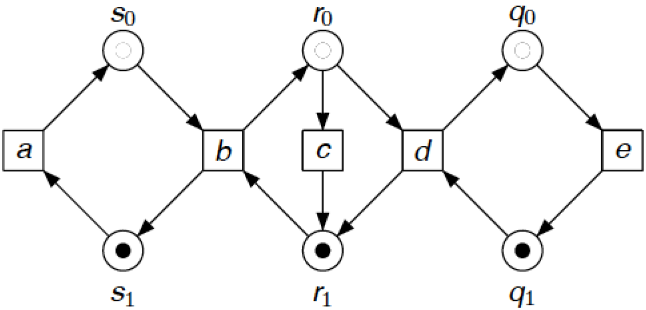

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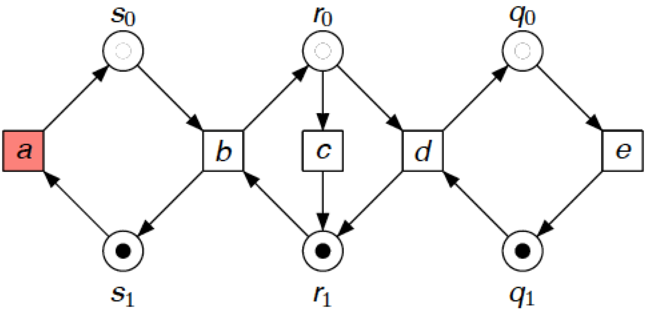

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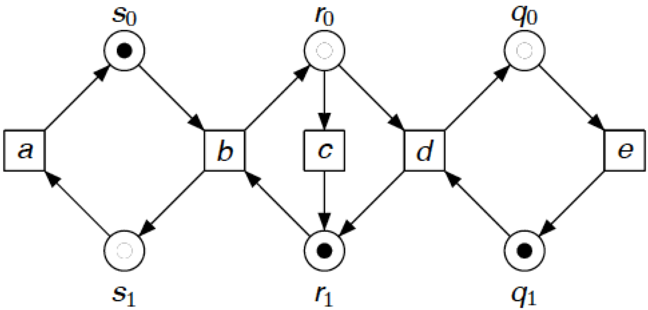

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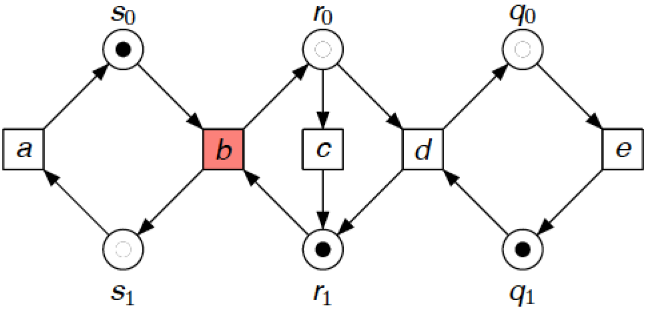
The interleaving semantics of Petri nets



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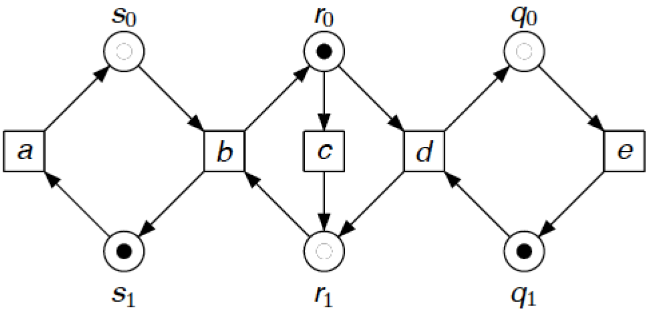
The interleaving semantics of Petri nets



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Reachable markings

Step sequence

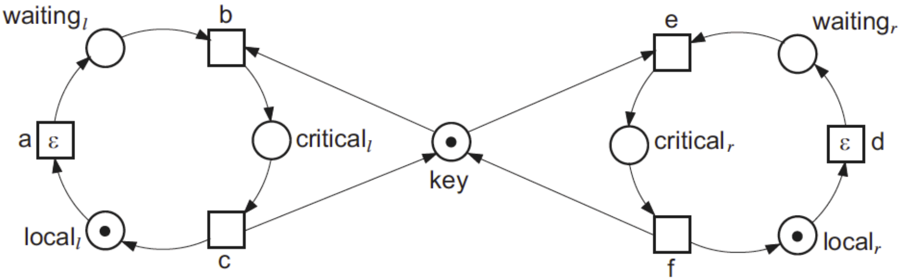
- A sequence of transitions $\sigma = t_1\, t_2 \ldots t_n$ is a step sequence if there exist markings M_1 through M_n such that:

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M_n$$

- Marking M_n is reached by the occurrence of σ, denoted $M_0 \xrightarrow{\sigma} M_n$.
- M is a reachable marking if there exists a step sequence σ with $M_0 \xrightarrow{\sigma} M$.

Mutual exclusion

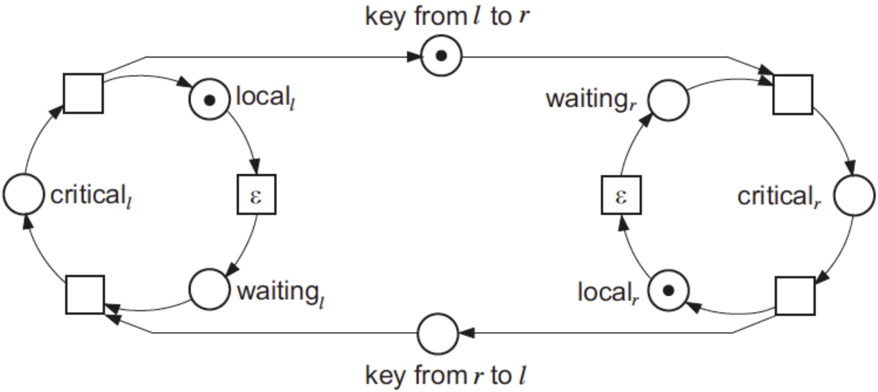
Two processes cycling through the states local, waiting and critical.



Between transitions b and e a conflict can arise infinitely often. No strategy has been modeled to solve this conflict.

Mutual exclusion

A strategy where processes are acquired access in an alternating fashion:



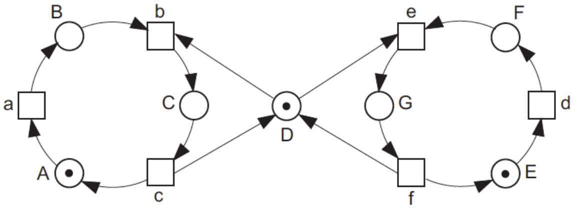
One-bounded elementary nets

1-bounded elementary net

An elementary net N is called 1-bounded if for each reachable marking M and place p of N:

$$M(p) \leqslant 1.$$

Markings of 1-bounded elementary nets can be described as a string of marked places, e.g., ADE. Two steps begin with this marking: $ADE \xrightarrow{a} BDE$ and $ADE \xrightarrow{d} ADF$.



Overview

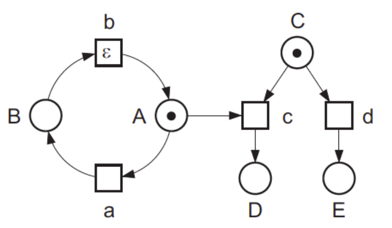
Sequential runs

Sequential run

Let N be an elementary net. A sequential run of N is a sequence

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots$$

of steps of N starting with the initial marking M_0. A run can be finite or infinite. A finite run $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_1} \cdots \xrightarrow{t_n} M_n$ is complete if M_n does not enable any transition, otherwise incomplete.



A sample complete run is:

$$AC \xrightarrow{a} BC \xrightarrow{b} AC \xrightarrow{c} D$$

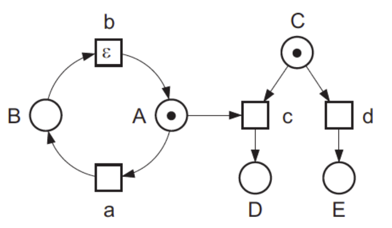
A sample incomplete run is:

$$AC \xrightarrow{d} AE \xrightarrow{a} BE$$

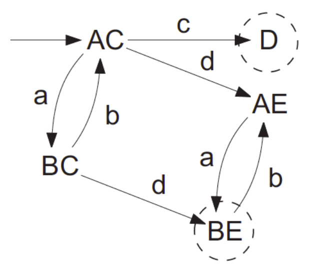
Marking graph

Marking graph

The marking graph of N has as nodes the reachable markings of N and as edges the reachable steps of N.



A sample elementary net



Its marking graph

Observation: 1-bounded nets induce finite marking graphs

Overview

Summary

- A Petri net consists of places, transitions and arcs
- An elementary net is a Petri net plus an initial marking
- Firing a single transition in a marking is a step
- A sequential run is a sequence of steps starting in the initial marking
- A marking graph has as nodes the reachable markings of the net and as edges its reachable steps
- The marking graph is the interleaving semantics of a net.