Static Program Analysis

Lecture 3: Dataflow Analysis II (Order-Theoretic Foundations)

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Outline

1. Recap: Dataflow Analysis
2. Heading for a Dataflow Analysis Framework
3. Order-Theoretic Foundations: The Domain

Labelled Programs

- Goal: localisation of analysis information
- Dataflow information will be associated with
  - `skip` statements
  - assignments
  - tests in conditionals (`if`) and loops (`while`)
- Assume set of labels Lab with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1\texttt{+}a_2 \mid a_1\texttt{-}a_2 \mid a_1\texttt{*}a_2 \in AExp \\
b &::= t \mid a_1\texttt{=}a_2 \mid a_1\texttt{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\texttt{skip}]^l \mid [x \;\texttt{:=}\; a]^l \mid c_1\,\texttt{;}\,c_2 \mid \\
&\qquad \texttt{if}\;[b]^l\;\texttt{then}\;c_1\;\texttt{else}\;c_2 \mid \texttt{while}\;[b]^l\;\texttt{do}\;c \in Cmd
\end{aligned}
$$

- All labels in $c \in Cmd$ assumed distinct, denoted by Lab_c
- Labelled fragments of c called blocks, denoted by Blk_c

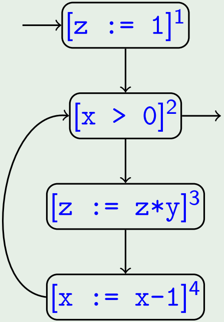
Representing Control Flow

Example

$c = $ `[z := 1]`^{1};
    `while [x > 0]`^{2} `do`
        `[z := z*y]`^{3};
        `[x := x-1]`^{4}

$$\begin{aligned} \mathsf{init}(c) &= 1 \\ \mathsf{final}(c) &= \{2\} \\ \mathsf{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\} \end{aligned}$$

Visualization by (control) flow graph:



Goal of Available Expressions Analysis

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The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

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- Can be used for Common Subexpression Elimination:
  replace subexpression by variable that contains up-to-date value
- Only interesting for non-trivial (i.e., complex) arithmetic expressions

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Example (Available Expressions Analysis)

```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
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- a+b available at label 3

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- a+b available at label 3
- a+b not available at label 5

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[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

- a+b available at label 3
- a+b not available at label 5
- possible optimization: `while [y > x]^3 do`

The Equation System I

- Analysis itself defined by setting up an equation system
- For each $l \in Lab_c$, $\mathsf{AE}_l \subseteq CExp_c$ represents the set of available expressions at the entry of block B^l
- Formally, for $c \in Cmd$ with isolated entry:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathsf{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l', l) \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

  where $\varphi_{l'} : 2^{CExp_c} \to 2^{CExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

- Characterization of analysis:

  flow-sensitive: results depending on order of assignments

  forward: starts in $\mathsf{init}(c)$ and proceeds downwards

  must: $\bigcap$ in equation for AE_l

- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathsf{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l', l) \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(E) = (E \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

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Example (AE equation system)

$c =$ [x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
[a := a+1]^4;
[x := a+b]^5

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$l \in Lab_c$	$\mathsf{kill}_{\mathsf{AE}}(B^l)$	$\mathsf{gen}_{\mathsf{AE}}(B^l)$
1	$\emptyset$	{a+b}
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3	$\emptyset$	{a+b}
4	{a+b, a*b, a+1}	$\emptyset$
5	$\emptyset$	{a+b}

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Equations:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}
\end{aligned}$$

The Equation System II

Reminder:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathrm{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l', l) \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(E) = (E \setminus \mathrm{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathrm{gen}_{\mathsf{AE}}(B^{l'})$$

Example (AE equation system)

$c = $
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[x := a+b]^1;
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  [a := a+1]^4;
  [x := a+b]^5
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$l \in Lab_c$	$\mathrm{kill}_{\mathsf{AE}}(B^l)$	$\mathrm{gen}_{\mathsf{AE}}(B^l)$
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\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Solution:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \{\texttt{a+b}\} \\
\mathsf{AE}_4 &= \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \emptyset
\end{aligned}$$

Goal of Live Variables Analysis

Live Variables Analysis

The goal of Live Variables Analysis is to determine, for each program point, which variables *may* be live at the exit from the point.

- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination:
  remove assignments to non-live variables

The Equation System I

- For each $l \in Lab_c$, $\mathsf{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l
- Formally, for a program $c \in Cmd$ with isolated exits:

$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l, l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

  where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

- Characterization of analysis:

  flow-sensitive: results depending on order of assignments

  backward: starts in $\mathsf{final}(c)$ and proceeds upwards

  may: $\bigcup$ in equation for LV_l

- Later: solution not necessarily unique

  $\Longrightarrow$ choose least one

The Equation System II

Reminder:

$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l, l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(V) = (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

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Example (LV equation system)

$c =$ [x := 2]^1;[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
[z := x]^5
else
[z := y*y]^6;
[x := z]^7

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Example (LV equation system)

$$\begin{aligned} c = {} & [\mathtt{x := 2}]^1; [\mathtt{y := 4}]^2; \\ & [\mathtt{x := 1}]^3; \\ & \mathtt{if}\ [\mathtt{y > 0}]^4\ \mathtt{then} \\ & \quad [\mathtt{z := x}]^5 \\ & \mathtt{else} \\ & \quad [\mathtt{z := y*y}]^6; \\ & [\mathtt{x := z}]^7 \end{aligned}$$

$l \in Lab_c$	$\mathsf{kill}_{\mathsf{LV}}(B^l)$	$\mathsf{gen}_{\mathsf{LV}}(B^l)$
1	$\{\mathtt{x}\}$	$\emptyset$
2	$\{\mathtt{y}\}$	$\emptyset$
3	$\{\mathtt{x}\}$	$\emptyset$
4	$\emptyset$	$\{\mathtt{y}\}$
5	$\{\mathtt{z}\}$	$\{\mathtt{x}\}$
6	$\{\mathtt{z}\}$	$\{\mathtt{y}\}$
7	$\{\mathtt{x}\}$	$\{\mathtt{z}\}$

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&= ((\mathsf{LV}_5 \setminus \{\mathtt{z}\}) \cup \{\mathtt{x}\}) \cup ((\mathsf{LV}_6 \setminus \{\mathtt{z}\}) \cup \{\mathtt{y}\}) \\
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\mathsf{LV}_7 &= \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}
\end{aligned}$$

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Reminder:

$$\mathrm{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathrm{final}(c) \\ \bigcup\{\varphi_{l'}(\mathrm{LV}_{l'}) \mid (l, l') \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

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Solution:

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- **Observation:** the analyses presented so far have some similarities

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- **Advantage:** possibility for designing (efficient) generic algorithms for solving dataflow equations
- **Overall pattern:** for $c \in Cmd$ and $l \in Lab_c$, the analysis information (AI) is described by equations of the form

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

  where

  - the set of extremal labels, E, is $\{\text{init}(c)\}$ or $\text{final}(c)$
  - ι specifies the extremal analysis information
  - the combination operator, $\bigsqcup$, is $\bigcap$ or $\bigcup$
  - $\varphi_{l'}$ denotes the transfer function of block $B^{l'}$
  - the flow relation F is $\text{flow}(c)$ or $\text{flow}^R(c)$ $(:= \{(l', l) \mid (l, l') \in \text{flow}(c)\})$

Characterization of Analyses

- **Direction of information flow:**
  - forward:
    - $F = \mathrm{flow}(c)$
    - AI_l concerns entry of B^l
    - c has isolated entry
  - backward:
    - $F = \mathrm{flow}^R(c)$
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  - backward:
    - $F = \text{flow}^R(c)$
    - AI_l concerns exit of B^l
    - c has isolated exits
- **Quantification over paths:**
  - may:
    - $\bigsqcup = \bigcup$
    - property satisfied by some path
    - interested in least solution (later)
  - must:
    - $\bigsqcup = \bigcap$
    - property satisfied by all paths
    - interested in greatest solution (later)

Roadmap

Goal: solve dataflow equation system by fixpoint iteration

1. Characterize solution of equation system as fixpoint of a transformation

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3. Establish least upper bound as combination operator
4. Ensure monotonicity of transfer functions
5. Guarantee termination of fixpoint iteration by ascending chain condition
6. Optimize fixpoint iteration by worklist algorithm

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- **Idea:** characterize solution as fixpoint of transformation:

$$(\mathsf{AI}_l = \tau_l)_{l \in Lab_c} \iff \Phi((\mathsf{AI}_l)_{l \in Lab_c}) = (\mathsf{AI}_l)_{l \in Lab_c}$$

  where $\Phi\left((\mathsf{AI}_l)_{l \in Lab_c}\right) := (\tau_l)_{l \in Lab_c}$

Motivation

- **Wanted:** solution of (dataflow) equation system
- **Problem:** recursive dependencies between dataflow variables
- **Idea:** characterize solution as fixpoint of transformation:

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  where $\Phi\left((\mathsf{AI}_l)_{l \in Lab_c}\right) := (\tau_l)_{l \in Lab_c}$
- **Approach:** approximate fixpoint by iteration

Partial Orders

The domain of analysis information usually forms a partial order where the ordering relation compares the "precision" of information.

Definition 3.1 (Partial order)

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

reflexivity: $d_1 \sqsubseteq d_1$

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Upper Bounds

In the dataflow equation system, analysis information from several predecessors is combined by taking the least upper bound.

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Complete Lattices

Since $\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\}$ can contain arbitrary elements, the existence of least upper bounds must be ensured for arbitrary subsets.

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Duality in Complete Lattices

- Dual concept of least upper bound: greatest lower bound
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- **Corollary:** every complete lattice has a greatest element $\top := \bigsqcap \emptyset$

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Chains are generated by the approximation of the analysis information in the fixpoint iteration.

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The Ascending Chain Condition I

Termination of fixpoint iteration is guaranteed by the following condition.

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- A sequence $(d_i)_{i\in\mathbb{N}}$ is called an ascending chain in D if $d_i \sqsubseteq d_{i+1}$ for each $i \in \mathbb{N}$.

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Notes:

- The finite height property implies ACC, but not vice versa (as there might be non-stabilizing descending chains)
- The complete lattice and ACC properties are orthogonal

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Domain requirements for dataflow analysis

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