Static Program Analysis

Lecture 2: Dataflow Analysis I
(Introduction & Available Expressions/Live Variables Analysis)

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http://moves.rwth-aachen.de/teaching/ws-1415/spa/

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Outline

Dataflow Analysis: the Approach

- Traditional form of program analysis

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- Idea: describe how analysis information flows through program

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- Traditional form of program analysis
- Idea: describe how analysis information flows through program
- Distinctions:

  dependence on statement order:
  : flow-sensitive vs. flow-insensitive analyses

  direction of flow:
  : forward vs. backward analyses

  quantification over paths:
  : may (union) vs. must (intersection) analyses

  procedures:
  : interprocedural vs. intraprocedural analyses

Labelled Programs

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  - `skip` statements
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Labelled Programs

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- Dataflow information will be associated with
  - `skip` statements
  - assignments
  - tests in conditionals (`if`) and loops (`while`)
- Assume set of labels Lab with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition 2.1 (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1\texttt{+}a_2 \mid a_1\texttt{-}a_2 \mid a_1\texttt{*}a_2 \in AExp\\
b &::= t \mid a_1\texttt{=}a_2 \mid a_1\texttt{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp\\
c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1\texttt{;}c_2 \mid\\
&\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } [b]^l \texttt{ do } c \in Cmd
\end{aligned}
$$

- All labels in $c \in Cmd$ assumed distinct, denoted by Lab_c
- Labelled fragments of c called blocks, denoted by Blk_c

A WHILE Program

Example 2.2

```
x := 6;
y := 7;
z := 0;
while x > 0 do
  x := x - 1;
  v := y;
  while v > 0 do
    v := v - 1;
    z := z + 1
```

A WHILE Program with Labels

Example 2.2

```
[x := 6]^1;
[y := 7]^2;
[z := 0]^3;
while [x > 0]^4 do
  [x := x - 1]^5;
  [v := y]^6;
  while [v > 0]^7 do
    [v := v - 1]^8;
    [z := z + 1]^9
```

Representing Control Flow I

Every (labelled) statement has a single entry (given by the initial label) and generally multiple exits (given by the final labels):

Definition 2.3 (Initial and final labels)

The mapping $\mathsf{init} : \mathit{Cmd} \to \mathit{Lab}$ returns the initial label of a statement:

$$\begin{aligned}
\mathsf{init}([\texttt{skip}]^l) &:= l \\
\mathsf{init}([x \texttt{ := } a]^l) &:= l \\
\mathsf{init}(c_1 \texttt{;} c_2) &:= \mathsf{init}(c_1) \\
\mathsf{init}(\texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2) &:= l \\
\mathsf{init}(\texttt{while } [b]^l \texttt{ do } c) &:= l
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\mathsf{init}(c_1 ; c_2) &:= \mathsf{init}(c_1) \\
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\mathsf{init}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= l
\end{aligned}$$

The mapping $\mathsf{final} : \mathit{Cmd} \to 2^{\mathit{Lab}}$ returns the set of final labels of a statement:

$$\begin{aligned}
\mathsf{final}([\mathtt{skip}]^l) &:= \{l\} \\
\mathsf{final}([x \mathrel{\mathtt{:=}} a]^l) &:= \{l\} \\
\mathsf{final}(c_1 ; c_2) &:= \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathsf{final}(c_1) \cup \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \{l\}
\end{aligned}$$

Representing Control Flow II

Definition 2.4 (Flow relation)

Given a statement $c \in Cmd$, the (control) flow relation

$$\mathrm{flow}(c) \subseteq Lab \times Lab$$

is defined by

$$\begin{aligned}
\mathrm{flow}([\mathtt{skip}]^l) &:= \emptyset \\
\mathrm{flow}([x \mathtt{\ :=\ } a]^l) &:= \emptyset \\
\mathrm{flow}(c_1 ; c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\quad\ \{(l, \mathrm{init}(c_2)) \mid l \in \mathrm{final}(c_1)\} \\
\mathrm{flow}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\quad\ \{(l, \mathrm{init}(c_1)), (l, \mathrm{init}(c_2))\} \\
\mathrm{flow}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \mathrm{flow}(c) \cup \{(l, \mathrm{init}(c))\} \cup \\
&\quad\ \{(l', l) \mid l' \in \mathrm{final}(c)\}
\end{aligned}$$

Representing Control Flow III

Example 2.5

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c = [z := 1]^1;
    while [x > 0]^2 do
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Representing Control Flow III

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$$\begin{aligned} \text{init}(c) &= 1 \\ \text{final}(c) &= \{2\} \\ \text{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\} \end{aligned}$$

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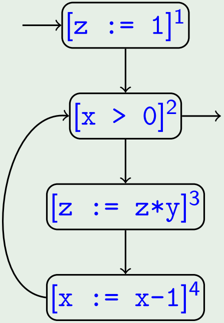
Visualization by (control) flow graph:



Representing Control Flow IV

- To simplify the presentation we will often assume that the program $c \in Cmd$ under consideration has an isolated entry, meaning that
$$\{l \in Lab \mid (l, \text{init}(c)) \in \text{flow}(c)\} = \emptyset$$
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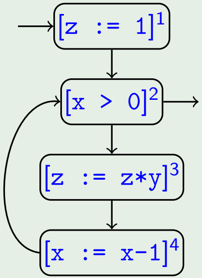
Example 2.6 (cf. Example 2.5)



has an isolated entry but not isolated exits

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Goal of Available Expressions Analysis

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The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

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```

- a+b available at label 3
- a+b not available at label 5
- possible optimization:
  `while [y > x]^3 do`

Formalizing Available Expressions Analysis I

- Given $a \in AExp$, $b \in BExp$, $c \in Cmd$
  - $Var_a/Var_b/Var_c$ denotes the set of all variables occurring in $a/b/c$
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- Formally: $\mathsf{kill}_{\mathsf{AE}} : Blk_c \to 2^{CExp_c}$ is defined by

$$\begin{aligned}
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$$\begin{aligned}\mathsf{gen}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset\\ \mathsf{gen}_{\mathsf{AE}}([x \ \mathtt{:=}\ a]^l) &:= \{a \mid x \notin Var_a\}\\ \mathsf{gen}_{\mathsf{AE}}([b]^l) &:= CExp_b\end{aligned}$$

Formalizing Available Expressions Analysis II

Example 2.8 ($\mathrm{kill}_{AE}/\mathrm{gen}_{AE}$ functions)

$c =$
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Lab_c	$\mathsf{kill}_{AE}(B^l)$	$\mathsf{gen}_{AE}(B^l)$
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3	$\emptyset$	$\{\texttt{a+b}\}$
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  where $\varphi_{l'} : 2^{CExp_c} \to 2^{CExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

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  forward: starts in $\mathrm{init}(c)$ and proceeds downwards

  must: $\bigcap$ in equation for AE_l
- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

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Equations:

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\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}
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&= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Solution:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \{\texttt{a+b}\} \\
\mathsf{AE}_4 &= \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \emptyset
\end{aligned}$$

Outline

Goal of Live Variables Analysis

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- All variables considered to be live at the end of the program (alternative: restriction to output variables)

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- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination: remove assignments to non-live variables

An Example

Example 2.10 (Live Variables Analysis)

```
[x := 2]^1;
[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
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else
  [z := y*y]^6;
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- y live at exit from 2
- x live at exit from 3

An Example

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[x := 2]1;
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- z live at exits from 5 and 6

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- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3
- z live at exits from 5 and 6
- possible optimization: remove $[\texttt{x := 2}]^1$

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- A variable on the left-hand side of an assignment is killed by the assignment; tests and `skip` do not kill

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$$\begin{aligned}
\mathsf{gen}_{\mathsf{LV}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathsf{gen}_{\mathsf{LV}}([x \mathtt{\ :=\ } a]^l) &:= Var_a \\
\mathsf{gen}_{\mathsf{LV}}([b]^l) &:= Var_b
\end{aligned}$$

Formalizing Live Variables Analysis II

Example 2.11 ($\mathsf{kill}_{LV}/\mathsf{gen}_{LV}$ functions)

$$
\begin{aligned}
c = {} & [\texttt{x := 2}]^1; \\
& [\texttt{y := 4}]^2; \\
& [\texttt{x := 1}]^3; \\
& \texttt{if } [\texttt{y > 0}]^4 \texttt{ then} \\
& \quad [\texttt{z := x}]^5 \\
& \texttt{else} \\
& \quad [\texttt{z := y*y}]^6; \\
& [\texttt{x := z}]^7
\end{aligned}
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Formalizing Live Variables Analysis II

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- $Var_c = \{x, y, z\}$

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$l \in Lab_c$	$\text{kill}_{\text{LV}}(B^l)$	$\text{gen}_{\text{LV}}(B^l)$
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4	$\emptyset$	$\{y\}$
5	$\{z\}$	$\{x\}$
6	$\{z\}$	$\{y\}$
7	$\{x\}$	$\{z\}$

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$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l, l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

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  backward: starts in $\mathsf{final}(c)$ and proceeds upwards

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- Characterization of analysis:

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  backward: starts in $\mathsf{final}(c)$ and proceeds upwards
  may: $\bigcup$ in equation for LV_l
- Later: solution not necessarily unique
  $\Longrightarrow$ choose least one

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Reminder:

$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l, l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

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$c =$
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$$\begin{aligned}
\mathsf{LV}_1 &= \varphi_2(\mathsf{LV}_2) = \mathsf{LV}_2 \setminus \{\mathtt{y}\} \\
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\mathsf{LV}_4 &= \varphi_5(\mathsf{LV}_5) \cup \varphi_6(\mathsf{LV}_6) \\
&= ((\mathsf{LV}_5 \setminus \{\mathtt{z}\}) \cup \{\mathtt{x}\}) \cup ((\mathsf{LV}_6 \setminus \{\mathtt{z}\}) \cup \{\mathtt{y}\}) \\
\mathsf{LV}_5 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\} \\
\mathsf{LV}_6 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\} \\
\mathsf{LV}_7 &= \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}
\end{aligned}$$

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Reminder:

$$\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l,l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(V) = (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

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$c = [\mathtt{x} := 2]^1; [\mathtt{y} := 4]^2;$
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\mathsf{LV}_7 &= \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}
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Solution:

$$\begin{aligned}
\mathsf{LV}_1 &= \emptyset \\
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