Static Program Analysis

Lecture 15: Abstract Interpretation V (Numerical & Predicate Abstraction)

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http://moves.rwth-aachen.de/teaching/ws-1415/spa/

Winter Semester 2014/15

Oral Exam in Static Program Analysis

- Options:
  - Thu 12 March
  - Tue 24 March
  - Thu 26 March
  - Wed 08 April
- Registration via `https://terminplaner2.dfn.de/foodle/Exam-Static-Program-Analysis-54991` (accessible through `http://moves.rwth-aachen.de/teaching/ws-1415/spa/`)

Outline

1 Overview of Numerical Abstraction Domains

2 Overview of Abstraction Refinement Using Predicates

3 Predicate Abstraction

4 Abstract Semantics for Predicate Abstraction

Non-Relational Abstraction Domains

In non-relational domains, abstract values are independently referring to single variables:

- Signs (cf. Example 11.3): $\mathrm{sgn}(x) = s$ ($x \in Var$, $s \in \{+, -, 0\}$)
- Intervals (cf. Example 11.4): $x \in J$
  ($x \in Var$, $J \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \cup \{\emptyset\}$)
- Parities (cf. Example 11.2): $x \in \mathbb{Z}_p$ ($x \in Var$, $p \in \{\mathrm{even}, \mathrm{odd}\}$)
- Congruences (cf. Lemma 14.4): $x \bmod m = k$
  ($x \in Var$, $m > 1$, $k \in \{0, \ldots, m-1\}$)

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Observations

- Expressive power:
  - Signs < Intervals (since $+ \cong [1, \infty]$, ...)
  - Parities < Congruences (since x even $\iff x \bmod 2 = 0$, ...)
  - Intervals and Congruences are incomparable
- Congruences can prove disequalities but not inequalities
  - e.g., $x \bmod m \neq y \bmod m \implies$ no zero division in $1/(x - y)$
- Mutual dependencies like $x \leq y$ generally not representable
- Non-relational domains efficient to represent and manipulate

Relational Abstraction Domains

In relational domains, interdependencies between variables are captured:

- Difference Bound Matrices (DBMs):
  conjunctions of $x - y \leq c$ and $\pm x \leq c$ ($x, y \in Var$, $c \in \mathbb{Z}$)
- Octagons: conjunctions of $ax + by \leq c$
  ($x, y \in Var$, $a, b \in \{-1, 0, 1\}$, $c \in \mathbb{Z}$)
- Octahedra: conjunctions of $a_1x_1 + \ldots + a_nx_n \leq c$
  ($x_i \in Var$, $a_i \in \{-1, 0, 1\}$, $c \in \mathbb{Z}$)
- Polyhedra: conjunctions of $a_1x_1 + \ldots + a_nx_n \leq c$
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Observations

- Expressive power:
  - DBMs < Octagons < Octahedra < Polyhedra
  - Intervals < DBMs (since $x \in [c_1, c_2] \iff -x \leq -c_1 \wedge x \leq c_2$)
- Can prove inequalities but not (general) disequalities
- Representation and manipulation generally more involved
  - Polyhedra require computation of convex hulls (exponential in $|Var|$)

Combining Non-Relational and Relational Domains

Linear Congruences combine features of Congruences and Polyhedra:

- given by conjunctions of

$$(a_1 x_1 + \ldots + a_n x_n) \bmod m = z$$

  ($x_i \in Var$, $a_i \in \mathbb{Z}$, $m > 1$, $z \in \mathbb{Z}$)
- typical application:

  $2x + 1 \bmod m \neq y \bmod m \implies$ no zero division in $1/(2x + 1 - y)$

- Again usable for proving disequalities but not inequalities

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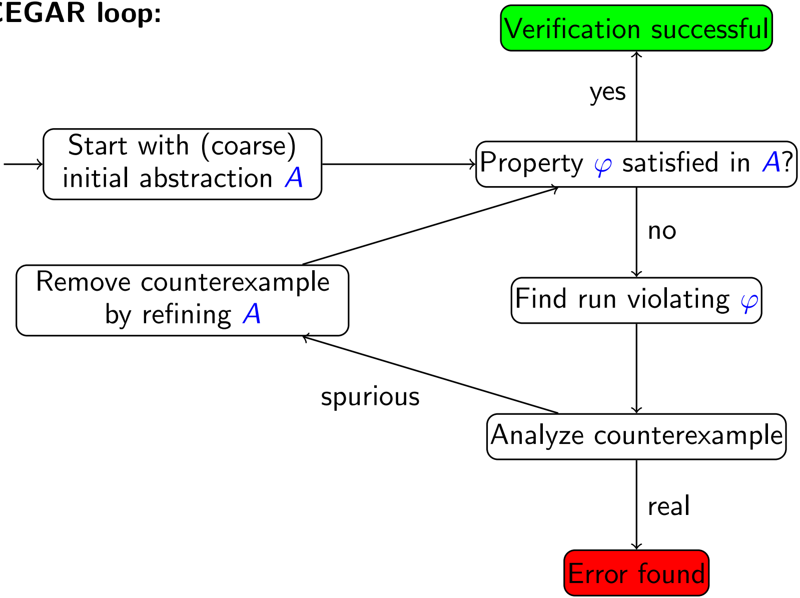
  - predicate abstraction and
  - analyzing counterexamples
- **Difference** to standard abstract interpretation: abstraction parametrised by and specific to program

Counterexample-Guided Abstraction Refinement

CEGAR loop:



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2. Use Galois connection that classifies program states according to validity of predicates (predicate abstraction)
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4. Iterate until property satisfied or real counterexample found (with increasing set of predicates)

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- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(p_1, \ldots, p_n) := \left(\left\{\bigwedge Q \mid Q \subseteq P \cup \neg P\right\}, \models\right).$$

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Abbreviations: $\mathsf{true} := \bigwedge \emptyset, \mathsf{false} := \bigwedge \{p_i, \neg p_i, \ldots\}$

Predicate Abstraction II

Lemma 15.2

$Abs(p_1, \ldots, p_n)$ *is a complete lattice with*

- $\bot = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = Q_1 \wedge Q_2$
- $Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2}$ *where* $\overline{b} := \bigwedge\{q \in P \cup \neg P \mid b \models q\}$
  (i.e., strongest formula in $Abs(p_1, \ldots, p_n)$ *that is implied by* $Q_1 \vee Q_2$*)*

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Example 15.3

Let $P := \{p_1, p_2, p_3\}$.

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain

$$\begin{aligned} Q_1 \sqcap Q_2 &= \underline{Q_1 \wedge Q_2} \equiv \underline{p_1 \wedge \neg p_2 \wedge p_3} \\ Q_1 \sqcup Q_2 &= \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} \equiv \neg p_2 \end{aligned}$$

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$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} \equiv \neg p_2$$
2. For $Q_1 := p_1 \wedge p_2$ and $Q_2 := p_1 \wedge \neg p_2$, we obtain
$$Q_1 \sqcap Q_2 = Q_1 \wedge Q_2 \equiv \text{false}$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{p_1 \wedge (p_2 \vee \neg p_2)} \equiv p_1$$

Predicate Abstraction III

Definition 15.4 (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\alpha : 2^{\Sigma} \to Abs(p_1, \ldots, p_n) \quad \text{and} \quad \gamma : Abs(p_1, \ldots, p_n) \to 2^{\Sigma}$$

with

$$\alpha(S) := \bigsqcup \{Q_\sigma \mid \sigma \in S\} \quad \text{and} \quad \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\}$$

where $Q_\sigma := \bigwedge(\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\})$.

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Example 15.5

- Let $Var := \{\mathtt{x}, \mathtt{y}\}$
- Let $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x<=y})$, $p_2 := (\mathtt{x=y})$, $p_3 := (\mathtt{x>y})$

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- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$, then

$$\begin{aligned} \alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= \underline{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)} \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$

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- If $Q = p_1 \wedge \neg p_2 \in Abs(p_1, \ldots, p_n)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

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Definition 15.6 (Execution relation for predicate abstraction)

If $c \in Cmd$ and $Q \in Abs(p_1, \ldots, p_n)$, then $\langle c, Q \rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, Q \rangle \Rightarrow \langle \downarrow, Q \rangle} \qquad \text{(asgn)}\ \frac{}{\langle x \texttt{ := } a, Q \rangle \Rightarrow \langle \downarrow, \bigsqcup\{Q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models Q\} \rangle}$$

$$\text{(seq1)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle c_1', Q' \rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_1' ; c_2, Q' \rangle} \qquad \text{(seq2)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle \downarrow, Q' \rangle}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_2, Q' \rangle}$$

$$\text{(if1)}\ \frac{}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, Q \rangle \Rightarrow \langle c_1, \overline{Q \wedge b} \rangle}$$

$$\text{(if2)}\ \frac{}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, Q \rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b} \rangle}$$

$$\text{(wh1)}\ \frac{}{\langle \texttt{while } b \texttt{ do } c, Q \rangle \Rightarrow \langle c ; \texttt{while } b \texttt{ do } c, \overline{Q \wedge b} \rangle}$$

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Remarks:

- In Rule (asgn), $\bigsqcup\{Q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models Q\}$ denotes the strongest postcondition of Q w.r.t. statement $x := a$. It covers all states that are obtained from a state satisfying Q by applying the assignment $x := a$:

$$\begin{array}{lccc} \text{Abstract:} & \langle x := a, Q\rangle & \Rightarrow & \langle\downarrow, \bigsqcup\{Q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models Q\}\rangle \\ & \downarrow\gamma & & \uparrow\alpha \\ \text{Concrete:} & \langle x := a, \{\sigma \in \Sigma \mid \sigma \models Q\}\rangle & \rightarrow & \langle\downarrow, \{\sigma[x \mapsto val_\sigma(a)] \mid \sigma \models Q\}\rangle \end{array}$$

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- An abstract configuration of the form $\langle c, \text{false}\rangle$ represents an unreachable configuration (as there is no $\sigma \in \Sigma$ such that $\sigma \models \text{false}$) and can therefore be omitted
- If $P = \emptyset$ (and thus $Abs(P) = \{\text{true}, \text{false}\}$) and if no $b \in BExp_c$ is a tautology or contradiction (i.e., resp. equivalent to true or false), then the abstract transition system corresponds to the control flow graph of c

Abstract Semantics for Predicate Abstraction III

Example 15.7

```
if [x > y]^1 then
  while [¬(y = 0)]^2 do
    [x := x - 1;]^3;
    [y := y - 1;]^4;
  if [x > y]^5 then
    [skip]^6;
  else
    [skip]^7;
else
  [skip]^8;
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- **Claim:** label 7 not reachable (as x > y is a loop invariant)
- **Proof:** by predicate abstraction with $p_1 := (x > y)$ and $p_2 := (x >= y)$
- **Abstract transition system:** on the board

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- **Proof:** by predicate abstraction with $p_1 := (\texttt{x > y})$ and $p_2 := (\texttt{x >= y})$
- **Abstract transition system:** on the board
- **Remark:** $p_1 := (\texttt{x > y})$ alone not sufficient to prove loop invariant (as not necessarily valid after label 3)