Exercise 1 (Partial Orders): (2 Points)

Consider the relation $\sqsubseteq \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ defined as

$$(a, b) \sqsubseteq (a', b') \quad :\Longleftrightarrow \quad a \cdot b' \leq a' \cdot b\ .$$

Prove or disprove: $(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \sqsubseteq)$ is a partial order.

Exercise 2 (Complete Lattices): (1 + 1.5 + 1.5 Points)

Let $(D, \sqsubseteq)$ be a complete lattice and let $\preceq \subseteq D \times D$ be defined as

$$d \preceq d' \quad :\Longleftrightarrow \quad d' \sqsubseteq d\ .$$

a) Prove that $(D, \preceq)$ is a complete lattice!

b) Let $S \subseteq D$ and let further $\sqcup_{\sqsubseteq} S$ denote the least upper bound of S with respect to the order $\sqsubseteq$ and let $\sqcap_{\preceq} S$ denote the greatest lower bound of S with respect to the order $\preceq$. Prove or disprove:

$$\forall S \subseteq D\colon\ \sqcup_{\sqsubseteq} S = \sqcap_{\preceq} S$$

c) Let $\bot_{\sqsubseteq}$ denote the least element of D with respect to the order $\sqsubseteq$ and let $\top_{\preceq}$ denote the greatest element of D with respect to the order $\preceq$. Prove or disprove:

$$\bot_{\sqsubseteq} = \top_{\preceq}$$

Exercise 3 (Monotinicity): (2 + 2 Points)

Let $(D, \sqsubseteq)$ be a complete lattice. We say that a function $f\colon D \to D$ is *supremum preserving* if for every ascending chain S we have that

$$f(\sqcup S) \quad = \quad \sqcup \{f(d) \mid d \in S\}$$

a) Prove or disprove: Every supremum preserving function is monotonic.

b) Prove or disprove: Every monotonic function is supremum preserving.