

**Theorem 1.** *Finite  $L \subseteq \text{Act}^*$  is realizable (by a weak CFM) if and only if  $L$  is closed under  $\models$ .*

$\implies$ . Assume  $L$  is realizable. Thus, there exists a weak CFM  $A$  (a CFM without synchronization messages) such that  $L = \text{Lin}(A)$ . As  $\text{Lin}(A)$  only contains linearizations, and every linearization is well-formed, each word in  $L$  is well formed. Let  $w \in \text{Act}^*$ , be well-formed, and assume  $L \models w$ . By definition of  $\models$ , this means that for every process  $p$  there exists a word  $v^p \in L$  such that  $v^p \upharpoonright p = w \upharpoonright p$ . We show that  $w \in L$ . (Then it follows that  $L$  is closed under  $\models$ .) This goes as follows.

Let  $\pi$  be an accepting run of CFM  $A$  on  $v^p$ . (Such run does exist, otherwise,  $v^p$  would not belong to  $L$ .) Let  $\pi \upharpoonright p$  be the projection of run  $\pi$  of  $A$  by only considering the transitions along  $\pi$  that take place at process  $p$ . Thus, transitions along  $\pi_p = \pi \upharpoonright p$  correspond to the "local" transitions of process  $p$ . It follows from  $v^p \in L$  that the local run  $\pi_p$  is an accepting run of local automaton (NFA)  $A_p$  (of process  $p$ ) on  $v^p \upharpoonright p$  (which equals  $w \upharpoonright p$ ). Here, accepting means that the local run  $\pi_p$  ends in a local accept state of  $A_p$ . This applies to all processes  $P = \{p_1, \dots, p_n\}$  of the CFM. The local accepting runs  $\pi_{p_1} = \pi \upharpoonright p_1, \dots, \pi_{p_n} = \pi \upharpoonright p_n$  can be combined to obtain a run,  $\pi^w$  say, of CFM  $A$  on  $w$  in a straightforward manner. The run  $\pi^w$  is accepting, as all processes end in a local accepting state, and all channels are empty, as  $\pi$  was accepting and  $\pi^w$  is constituted from "bits" spanning  $\pi$ . Thus there cannot be "open" receipts. Thus  $w \in L$ .

$\impliedby$ . Assume  $L$  is closed under  $\models$ . As  $\models$  is only defined for well-formed words, each word in  $L$  is well formed. Moreover, by definition of closure under  $\models$ ,  $L \models w$  implies  $w \in L$  for each well-formed  $w \in \text{Act}^*$ . Proof obligation:  $L$  is realizable. This goes as follows:

Let  $A_p$  be an NFA over the alphabet  $\text{Act}_p$  accepting  $L_p = \{w \upharpoonright p \mid w \in L\}$ . As  $L$  is finite,  $L_p$  is finite and regular, thus  $A_p$  is indeed an NFA.  $A_p$  thus accepts all projections to process  $p$  of words in  $L$ . Let weak CFM  $A = ((A_p)_{p \in P}, s_{\text{init}}, F)$  with  $F = \prod_{p \in P} F_p$ . We now claim that  $A$  is a realization of  $L$ , i.e.,  $\text{Lin}(A) = L$ . This claim can be proven as follows:

- $\supseteq$  Let  $w \in L$ . By construction of CFM  $A$ ,  $\text{Lin}(A_p) = L_p$ . But then  $w \in \text{Lin}(A)$ .
- $\subseteq$  Let  $w \in \text{Lin}(A)$ . Then  $w \upharpoonright p \in \text{Lin}(A_p)$  for each  $p \in P$ . By definition of  $\models$ , it follows  $L \models w$ . Since  $L$  is closed under  $\models$ , it follows  $w \in L$ .