Towards Delays in Dynamical and Control Systems

Simulation-Based Verification & Game Theory

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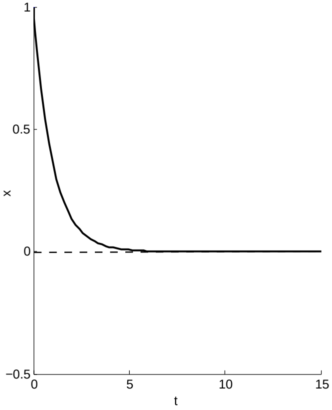
Outline

Outline

Delayed Differential Dynamics

$$\begin{cases} \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases}$$



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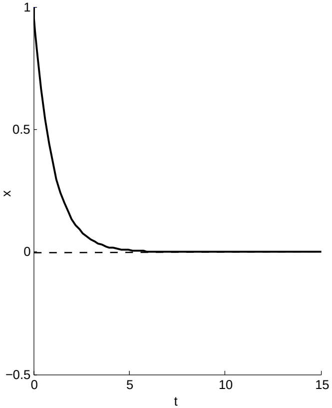
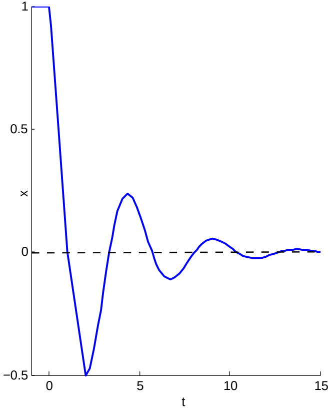
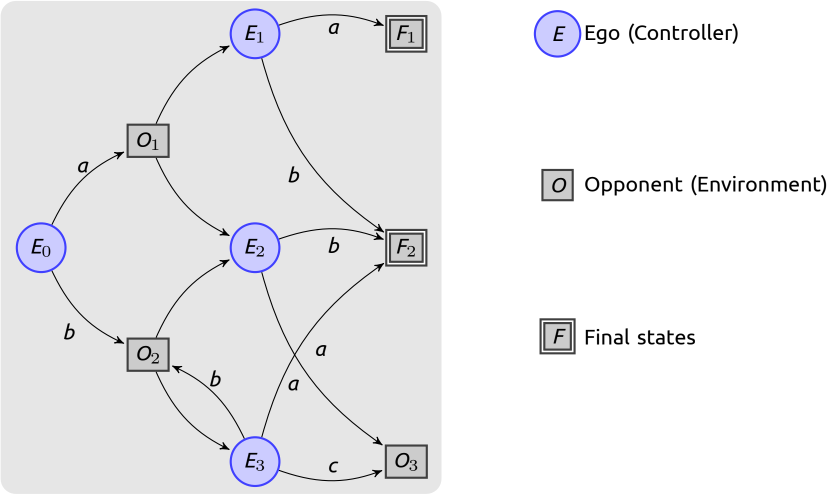


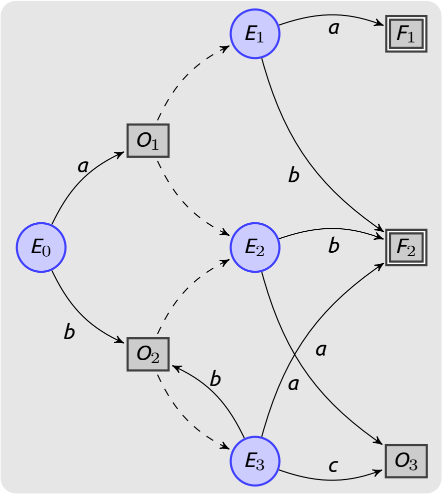
$$\begin{cases} \dot{x}(t) = -x(t-1) \\ x([-1,0]) \equiv 1 \end{cases}$$

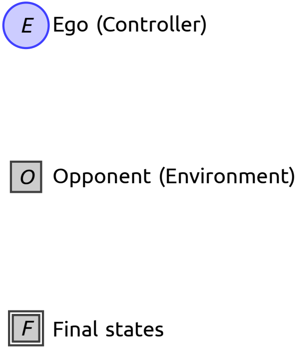


Games with Perfect Information



Games with Delayed Information





Outline

Delayed Dynamical Systems

Delayed Dynamical Systems

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \boldsymbol{f}(\mathbf{x}(t), \mathbf{x}(t-r_1), \ldots, \mathbf{x}(t-r_k)), \quad t \in [0, \infty) \\ \mathbf{x}(t) & \equiv & \mathbf{x}_0 \in \Theta, \quad t \in [-r_k, 0] \end{cases}$$

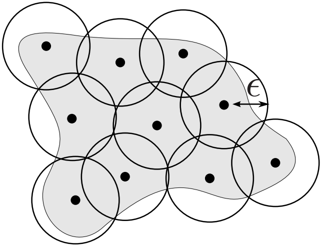
The unique *solution* (*trajectory*) : $\xi_{\mathbf{x}_0}(t) : [-r_k, \infty) \mapsto \mathbb{R}^n$.

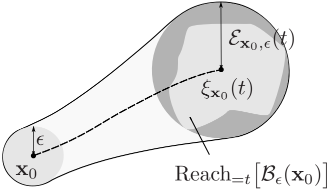
Safety Verification Problem

Given $T \in \mathbb{R}$, $\mathcal{X}_0 \subseteq \Theta$, $\mathcal{U} \subseteq \mathbb{R}^n$, weather

$$\forall \mathbf{x}_0 \in \mathcal{X}_0 : \quad \left(\bigcup_{t \leq T} \xi_{\mathbf{x}_0}(t) \right) \cap \mathcal{U} = \emptyset \quad ?$$

Basic Idea [1]





1. Figures are taken from [A. DonzDonzé and O. Maler, HSCC'07].

Verification Algorithm

Algorithm 1: Simulation-based Verification for Delayed Dynamical Systems

input : The dynamics $\boldsymbol{f}(\mathbf{x}, \mathbf{u})$, delay term r, initial set $\mathcal{X}_0$, unsafe set $\mathcal{U}$, time bound T, precision ϵ.

`/* initialization */`

$S \leftarrow \mathcal{X}_0;\ \mathcal{R} \leftarrow \emptyset;\ \delta \leftarrow dia(\mathcal{X}_0)/2;\ \tau \leftarrow \tau_0;$
while $S \neq \emptyset$ **do**
    $\mathcal{X} \leftarrow \delta$-Partition$(S)$;
    if $\delta < \epsilon$ **then**
        return (UNKNOWN, $\mathcal{R}$);
    for $\mathcal{B}_\delta(\mathbf{x}_0) \in \mathcal{X}$ **do**
        $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle \leftarrow$ Simulation$(\mathcal{B}_\delta(\mathbf{x}_0), \boldsymbol{f}(\mathbf{x}, \mathbf{u}), r, \tau, T)$;
        $\mathcal{T} \leftarrow \bigcup_{n=0}^{N-1} conv(B_{\mathbf{d}_n}(\mathbf{y}_n) \cup B_{\mathbf{d}_{n+1}}(\mathbf{y}_{n+1}))$;
        if $\mathcal{T} \cap \mathcal{U} = \emptyset$ **then**
            $S \leftarrow S \backslash \mathcal{B}_\delta(\mathbf{x}_0);\quad \mathcal{R} \leftarrow \mathcal{R} \cup \mathcal{T}$;
        else if $\exists i.\ \mathcal{B}_{\mathbf{d}_i}(\mathbf{y}_i) \subseteq \mathcal{U}$ **then**
            return (UNSAFE, $\mathcal{T}$);
        else
            $S \leftarrow S \backslash \mathcal{B}_\delta(\mathbf{x}_0);\quad \delta \leftarrow \delta/2$;
return (SAFE, $\mathcal{R}$);

Local Error Bounds

$$E(t) = \begin{cases} d_0, & \text{if } t = 0, \\ E(t_i) + (t - t_i)e_{i+1}, & \text{if } t \in [t_i, t_{i+1}]. \end{cases}$$

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Validation Property :

$$\xi_{\mathbf{x}_0}(t) \in \mathcal{B}_{E(t)}\left(\frac{(t - t_i)\mathbf{y}_i + (t_{i+1} - t)\mathbf{y}_{i+1}}{t_{i+1} - t_i}\right), \text{ for each } t \in [t_i, t_{i+1}].$$

Simulation Algorithm

Algorithm 2: Simulation: a validated DDE solver producing rigorous bounds

input : The initial set $\mathcal{B}_\delta(\mathbf{x}_0)$, dynamics $\boldsymbol{f}(\mathbf{x}, \mathbf{u})$, delay term r, step size τ, time bound T.

output: A triple $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle$, where the components represent lists, with the same length, respectively for the time points, numerical approximations (possibly multi-dimensional), and the rigorous local error bounds.

```
/* initializing the lists, whose indices start from -1 */
```
$\mathbf{t} \leftarrow [\![-\tau, 0]\!]$; $\mathbf{y} \leftarrow [\![\mathbf{x}_0, \mathbf{x}_0]\!]$; $\mathbf{d} \leftarrow [\![0, \delta]\!]$;

```
/* r has to be divisible by τ (in FP numbers) */
```
$n \leftarrow 0$; $m \leftarrow r/\tau$;

while $\mathbf{t}_n < T$ **do**

- $t_{n+1} \leftarrow \mathbf{t}_n + \tau$;
- `/* approximating` y_{n+1} `using` *`forward Euler method`* `*/`
- $y_{n+1} \leftarrow \mathbf{y}_n + \boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m}) * \tau$;
- `/* computing error slope by constrained optimization */`
- $e_n \leftarrow$ **Find** minimum e s.t.

$$\begin{cases} \|\boldsymbol{f}(\mathbf{x} + t * \mathbf{f}, \mathbf{u} + t * \mathbf{g}) - \boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m})\| \leq e - \sigma, \text{ for} \\ \forall t \in [0, \tau] \\ \forall \mathbf{x} \in \mathcal{B}_{\mathbf{d}_n}(\mathbf{y}_n) \\ \forall \mathbf{u} \in \mathcal{B}_{\mathbf{d}_{n-m}}(\mathbf{y}_{n-m}) \\ \forall \mathbf{f} \in \mathcal{B}_e(\boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m})) \\ \forall \mathbf{g} \in \mathcal{B}_{e_{n-m}}(\boldsymbol{f}(\mathbf{y}_{n-m}, \mathbf{y}_{n-2m})); \end{cases} \tag{9}$$

- $d_{n+1} \leftarrow \mathbf{d}_n + \tau e_n$;
- `/* updating the lists by appending the extrapolation */`
- $\mathbf{t} \leftarrow [\![\mathbf{t}, t_{n+1}]\!]$; $\mathbf{y} \leftarrow [\![\mathbf{y}, y_{n+1}]\!]$; $\mathbf{d} \leftarrow [\![\mathbf{d}, d_{n+1}]\!]$;
- $n \leftarrow n + 1$;

return $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle$;

Solving the Optimization by HySAT-II

$$\text{find } \min\{e \geq 0 \mid \forall x : \phi(x, e) \implies \psi(x, e)\}$$

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$$\text{find } \max\{e \geq 0 \mid \exists x : \phi(x, e) \wedge \neg\psi(x, e)\}$$

Simulation Algorithm

Theorem (Correctness)

Suppose the maximum index of the lists is N, then $\forall t \in [0, T]$ *and* $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$,

$$\xi_{\mathbf{x}}(t) \subseteq \bigcup_{n=0}^{N-1} conv(\mathcal{B}_{\mathbf{d}_n}(\mathbf{y}_n) \cup \mathcal{B}_{\mathbf{d}_{n+1}}(\mathbf{y}_{n+1})).$$

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Theorem (Completeness)

Suppose the function $\boldsymbol{f}$ *is continuously differentiable in both arguments and the dynamical system is solvable for time interval* $[0, T]$, *then for any* $\varepsilon > 0$, *there exists* δ, τ *and* σ *such that the optimization problem (9) has a solution* e_n *for all* $n \leq \frac{T}{\tau}$, *and moreover* $\mathbf{d}_n \leq \varepsilon$.

Simulation Algorithm

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Further extension to simulations with variable stepsize.

Delayed Logistic Equation

$$\dot{N}(t) = N(t)[1 - N(t - r)]$$

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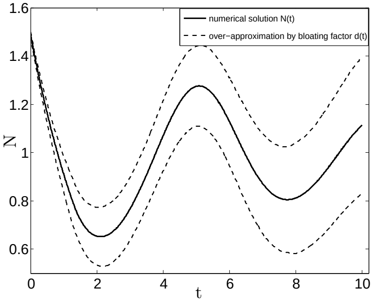

Figure : $\mathcal{X}_0 = \mathcal{B}_{0.01}(1.49)$, $r = 1.3$, $\tau_0 = 0.01$, $T = 10$s.

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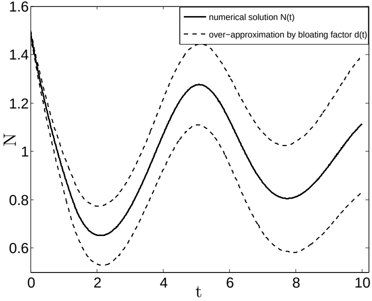

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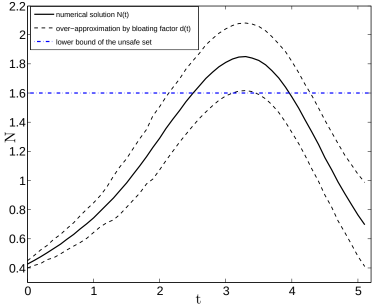
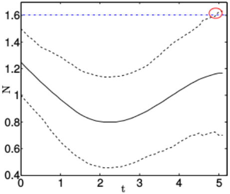
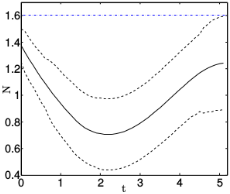


Figure : Over-approximation rigorously proving unsafe, with $r = 1.7$, $\mathcal{X}_0 = \mathcal{B}_{0.025}(0.425)$, $\tau_0 = 0.1$, $T = 5$s, $\mathcal{U} = \{N | N > 1.6\}$.

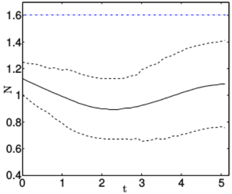
Delayed Logistic Equation



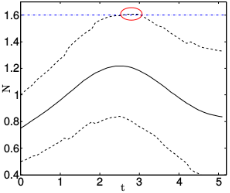
(a) An initial over-approximaion of trajectories starting from $\mathcal{B}_{0.225}(1.25)$. It overlaps with the unsafe set (s. circle). Initial set is consequently split (cf. Figs. 3b, 3c).



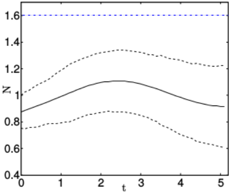
(b) All trajectories starting from $\mathcal{B}_{0.125}(1.375)$ are proven safe within the time bound, as the over-approximation does not intersect with the unsafe set.



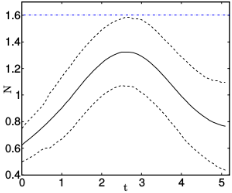
(c) Initial state set $\mathcal{B}_{0.125}(1.125)$ is verified to be safe as well.



(d) $\mathcal{B}_{0.25}(0.75)$ yields overlap w. unsafe; the ball is partitioned again (Figs. 3e, 3f).



(e) All trajectories originating from $\mathcal{B}_{0.125}(0.875)$ are provably safe.

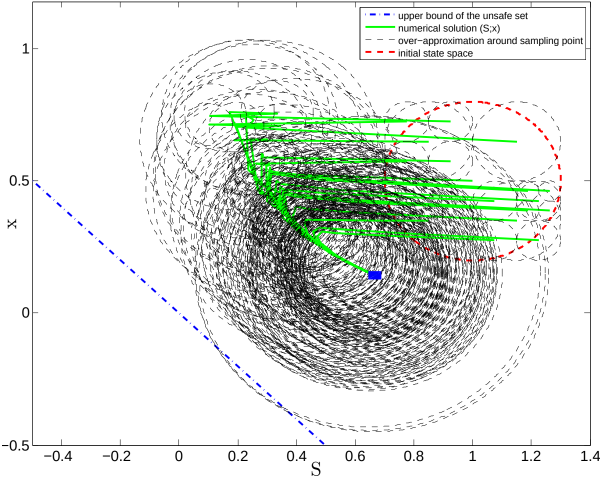


(f) All trajectories originating from $\mathcal{B}_{0.125}(0.625)$ are provably safe as well.

Fig. 3: The logistic system (13) is proven safe through 6 rounds of simulation with base stepsize $\tau_0 = 0.1$. Delay $r = 1.3$, initial state set $\mathcal{X}_0 = \{N | N \in [0.5, 1.5]\}$, time bound $T = 5$s, unsafe set $\{N | N > 1.6\}$.

Delayed Microbial Growth

$$\begin{cases} \dot{S}(t) = 1 - S(t) - f(S(t))x(t) \\ \dot{x}(t) = e^{-r} f(S(t-r))x(t-r) - x(t) \end{cases}$$



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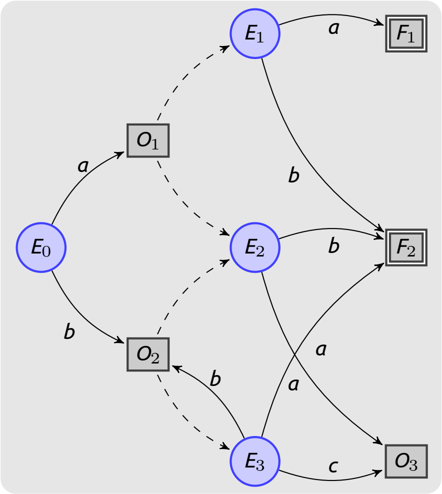
Observation-Based Subset-Construction

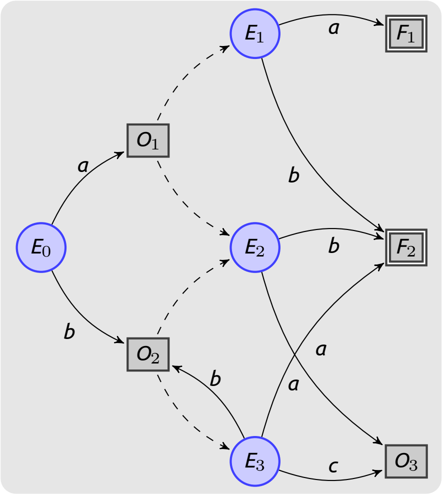


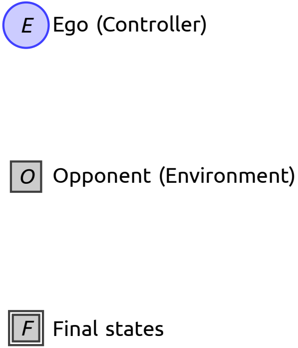
Ego (Controller)

Opponent (Environment)

Final states

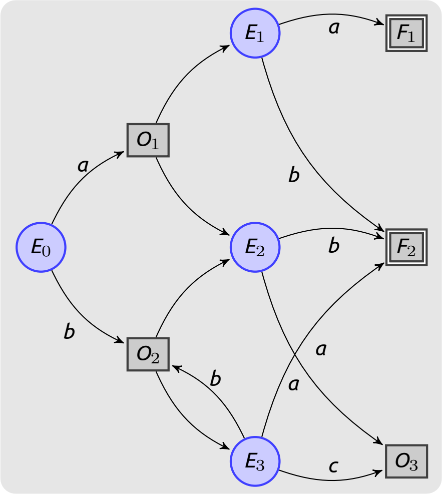
Observation-Based Subset-Construction



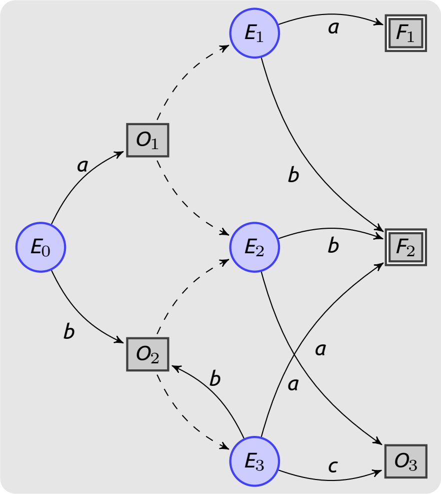


Is it possible to avoid the explosion of the state space ?

Lift the Step of Delays from $0, 1, \ldots, k$ by Labelling

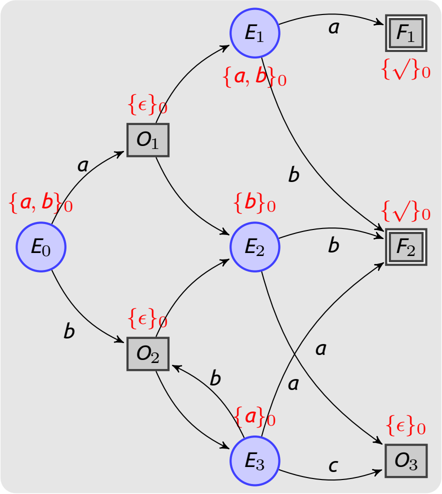


$delay = 0$

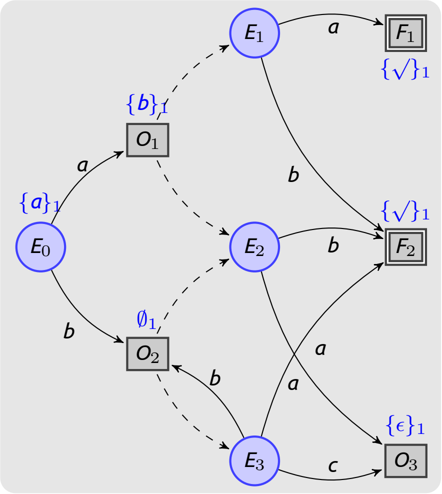


$delay = 1$

Lift the Step of Delays from $0, 1, \ldots, k$ by Labelling



$delay = 0$



$delay = 1$

Outline

Concluding Remarks

- Verifying delayed differential dynamics by validated simulation : a combination of numerical methods with SMT solvers.
- A sketch idea on synthesizing controllers in games of delayed information ; non-deterministic strategy & almost winning ?