Semantics and Verification of Software

Summer Semester 2019

Lecture 9: Axiomatic Semantics of WHILE I (Hoare Logic)

Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

`https://moves.rwth-aachen.de/teaching/ss-19/sv-sw/`

Outline of Lecture 9

Recap: Equivalence of Operational and Denotational Semantics

Equivalence of Semantics I

Remember: in Definition 4.1, $\mathfrak{O}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$ was given by

$$\mathfrak{O}[\![c]\!](\sigma) = \sigma' \iff \langle c, \sigma \rangle \to \sigma'$$

Theorem (Coincidence Theorem)

For every $c \in \mathit{Cmd}$,

$$\mathfrak{O}[\![c]\!] = \mathfrak{C}[\![c]\!],$$

i.e., $\langle c, \sigma \rangle \to \sigma'$ *iff* $\mathfrak{C}[\![c]\!](\sigma) = \sigma'$, *and thus* $\mathfrak{O}[\![.]\!] = \mathfrak{C}[\![.]\!]$.

Equivalence of Semantics II

Proof (Theorem 8.5).

We have to show that

$$\langle c, \sigma \rangle \to \sigma' \iff \mathfrak{C}[\![c]\!](\sigma) = \sigma'$$

- $\Rightarrow$ by structural induction over the derivation tree of $\langle c, \sigma \rangle \to \sigma'$ (as seen)
- $\Leftarrow$ by structural induction over c. We only consider $c = \texttt{while}\ b\ \texttt{do}\ c_0\ \texttt{end}$. Since $\mathfrak{C}[\![c]\!](\sigma) = \sigma'$, according to Cor. 8.3 there exists $n \geq 1$ such that $\Phi^n(f_\emptyset)(\sigma) = \sigma'$ where $\Phi(f) = \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c_0]\!], \mathrm{id}_\Sigma)$. We show $\langle c, \sigma \rangle \to \sigma'$ by complete induction on n:

  $n = 1$: (note that $n = 0$ is excluded as $\Phi^n(f_\emptyset)(\sigma)$ always undefined)

  $\Phi(f_\emptyset)(\sigma) = \sigma'$

  $\Rightarrow \mathfrak{B}[\![b]\!]\sigma = \mathrm{false}$ and $\sigma' = \sigma$ (def. Φ, $\mathrm{graph}(f_\emptyset) = \emptyset$)

  $\Rightarrow \langle b, \sigma \rangle \to \mathrm{false}$ (Lemma 8.6(2))

  $\Rightarrow \langle c, \sigma \rangle \to \sigma = \sigma'$ (rule (wh-f))

Equivalence of Semantics III

Proof (Theorem 8.5; continued).

[remember: $\Phi(f) = \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c_0]\!], \mathrm{id}_\Sigma)$]

$n \rightsquigarrow n+1$: here, $\Phi^{n+1}(f_\emptyset)(\sigma) = \Phi^n(\Phi(f_\emptyset))(\sigma) = \sigma'$.

The following cases are possible:

- $\mathfrak{B}[\![b]\!]\sigma = \mathrm{true}$: due to the definition of Φ, $\sigma' = \Phi^n(f_\emptyset)(\sigma'')$ for $\sigma'' := \mathfrak{C}[\![c_0]\!]\sigma$. Moreover,
  - $\langle b, \sigma\rangle \to \mathrm{true}$ (Lemma 8.6(2))
  - $\langle c_0, \sigma\rangle \to \sigma''$ (induction hypothesis for c_0)
  - $\langle c, \sigma''\rangle \to \sigma'$ (induction hypothesis for n)

  such that $\langle c, \sigma\rangle \to \sigma'$ (rule (wh-t)).
- $\mathfrak{B}[\![b]\!]\sigma = \mathrm{false}$: due to the definition of Φ, $\sigma' = \sigma$. Moreover, $\langle b, \sigma\rangle \to \mathrm{false}$ (Lemma 8.6(2)) such that $\langle c, \sigma\rangle \to \sigma = \sigma'$ (rule (wh-f)).

□

Outline of Lecture 9

The Axiomatic Approach I

Example 9.1

- Let $c \in \mathit{Cmd}$ be given by

```
s := 0; n := 1; while ¬(n>N) do s := s+n; n := n+1 end
```

The Axiomatic Approach I

Example 9.1

- Let $c \in \mathit{Cmd}$ be given by

$$\texttt{s := 0; n := 1; while } \neg\texttt{(n>N) do s := s+n; n := n+1 end}$$

- How to show that, after termination of c,

$$\sigma(\mathtt{s}) = \sum_{k=1}^{\sigma(\mathtt{N})} k \quad ?$$

The Axiomatic Approach I

Example 9.1

- Let $c \in \mathit{Cmd}$ be given by

$$\texttt{s := 0; n := 1; while } \neg\texttt{(n>N) do s := s+n; n := n+1 end}$$

- How to show that, after termination of c,

$$\sigma(\mathtt{s}) = \sum_{k=1}^{\sigma(\mathtt{N})} k \quad ?$$

- "Running" c according to the operational semantics in insufficient: every change of $\sigma(\mathtt{N})$ requires a new proof

The Axiomatic Approach I

Example 9.1

- Let $c \in \mathit{Cmd}$ be given by

$$\texttt{s := 0; n := 1; while } \neg\texttt{(n>N) do s := s+n; n := n+1 end}$$

- How to show that, after termination of c,

$$\sigma(\mathtt{s}) = \sum_{k=1}^{\sigma(\mathtt{N})} k \quad ?$$

- "Running" c according to the operational semantics in insufficient: every change of $\sigma(\mathtt{N})$ requires a new proof
- Wanted: a more abstract, "symbolic" way of reasoning

The Axiomatic Approach II

Example 9.1 (continued)

Obviously c satisfies the following assertions (after execution of the respective statement):

$$
\begin{array}{l}
\texttt{s := 0;} \\
\{\texttt{s} = 0\} \\
\texttt{n := 1;} \\
\{\texttt{s} = 0 \wedge \texttt{n} = 1\} \\
\texttt{while } \neg(\texttt{n>N}) \texttt{ do s := s+n; n := n+1 end} \\
\{\texttt{s} = \sum_{k=1}^{\texttt{N}} k \wedge \texttt{n} > \texttt{N}\}
\end{array}
$$

where, e.g., "$\texttt{s} = 0$" means "$\sigma(\texttt{s}) = 0$ in the current state $\sigma \in \Sigma$"

The Axiomatic Approach III

How to prove the validity of assertions?

- Assertions following assignments are evident ("$s = 0$")

The Axiomatic Approach III

How to prove the validity of assertions?

- Assertions following assignments are evident ("$\mathtt{s} = 0$")
- Also, "$\mathtt{n} > \mathtt{N}$" follows directly from the loop's execution condition

The Axiomatic Approach III

How to prove the validity of assertions?

- Assertions following assignments are evident ("$\mathtt{s} = 0$")
- Also, "$\mathtt{n} > \mathtt{N}$" follows directly from the loop's execution condition
- But how to obtain the final value of $\mathtt{s}$?

The Axiomatic Approach III

How to prove the validity of assertions?

- Assertions following assignments are evident ("$\mathtt{s} = 0$")
- Also, "$\mathtt{n} > \mathtt{N}$" follows directly from the loop's execution condition
- But how to obtain the final value of $\mathtt{s}$?
- Answer: after every loop iteration, the invariant $\mathtt{s} = \sum_{k=1}^{\mathtt{n}-1} k$ is satisfied

The Axiomatic Approach III

How to prove the validity of assertions?

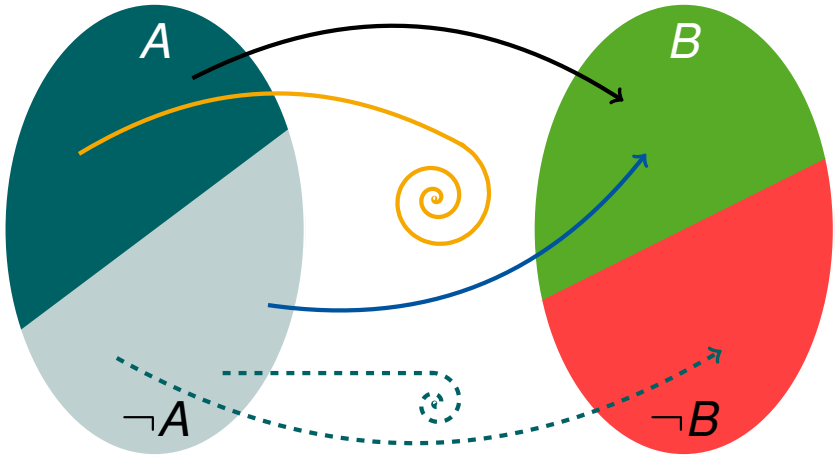
- Assertions following assignments are evident ("$\mathtt{s} = 0$")
- Also, "$\mathtt{n} > \mathtt{N}$" follows directly from the loop's execution condition
- But how to obtain the final value of $\mathtt{s}$?
- Answer: after every loop iteration, the invariant $\mathtt{s} = \sum_{k=1}^{\mathtt{n}-1} k$ is satisfied
- Our proof system employs partial correctness properties of the form $\{A\}\, c\, \{B\}$ with assertions A, B and $c \in \mathit{Cmd}$

The Axiomatic Approach III

How to prove the validity of assertions?

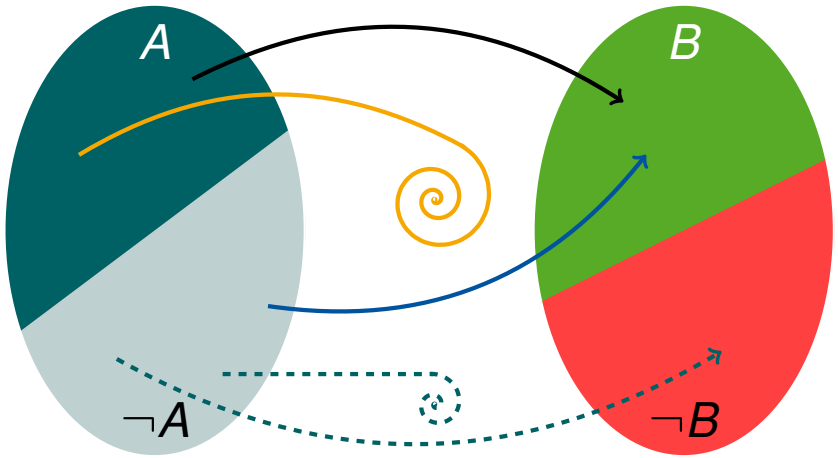
- Assertions following assignments are evident ("$\mathtt{s} = 0$")
- Also, "$\mathtt{n} > \mathtt{N}$" follows directly from the loop's execution condition
- But how to obtain the final value of $\mathtt{s}$?
- Answer: after every loop iteration, the invariant $\mathtt{s} = \sum_{k=1}^{\mathtt{n}-1} k$ is satisfied
- Our proof system employs partial correctness properties of the form $\{A\}\, c\, \{B\}$ with assertions A, B and $c \in \mathit{Cmd}$
- Interpretation depends on expected termination behaviour of c:
  partial correctness: nothing is said about c if it fails to terminate
  total correctness: c terminates on all inputs satisfying A

The Axiomatic Approach IV



(picture courtesy of C. Matheja)

The Axiomatic Approach IV



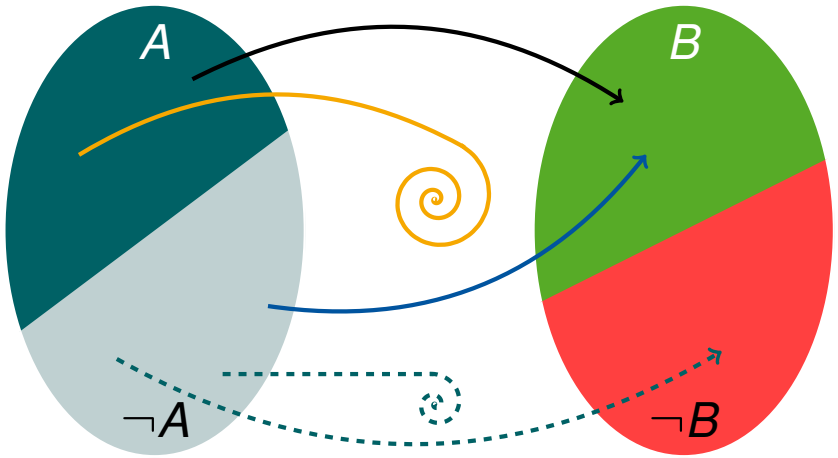
(picture courtesy of C. Matheja)

Partial correctness properties

$\{A\}\ c\ \{B\}$ is valid if for all states $\sigma \in \Sigma$ which satisfy A:
if the execution of c in σ terminates in $\sigma' \in \Sigma$, then σ' satisfies B.

In particular, $\{\text{true}\}\ c\ \{\text{false}\}$ is valid for $c = \texttt{while true do skip end}$

The Axiomatic Approach IV



(picture courtesy of C. Matheja)

Partial correctness properties

$\{A\}\ c\ \{B\}$ is valid if for all states $\sigma \in \Sigma$ which satisfy A:
if the execution of c in σ terminates in $\sigma' \in \Sigma$, then σ' satisfies B.

In particular, $\{\text{true}\}\ c\ \{\text{false}\}$ is valid for $c = \texttt{while true do skip end}$

Total correctness properties

$\{A\}\ c\ \{\Downarrow B\}$ is valid if for all states $\sigma \in \Sigma$ which satisfy A:
the execution of c in σ terminates and yields a state which satisfies B.

Outline of Lecture 9

Syntax of the Assertion Language I

Assertions = Boolean expressions + logical variables

- to memorize previous values of program variables
- to formulate more involved state properties

Syntax of the Assertion Language I

Assertions = Boolean expressions + logical variables

- to memorize previous values of program variables
- to formulate more involved state properties

Syntactic categories:

Category	Domain	Meta variable(s)
Logical variables	*LVar*	i
Arithmetic expressions with logical variables	*LExp*	a
Assertions	*Assn*	A, B, C

Syntax of the Assertion Language II

Definition 9.2 (Syntax of assertions)

The syntax of *Assn* is defined by the following context-free grammar:

$$a ::= z \mid x \mid i \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 * a_2 \in \mathit{LExp}$$
$$A ::= t \mid a_1 = a_2 \mid a_1 > a_2 \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \forall i.A \in \mathit{Assn}$$

Syntax of the Assertion Language II

Definition 9.2 (Syntax of assertions)

The syntax of *Assn* is defined by the following context-free grammar:

$$a ::= z \mid x \mid i \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in \mathit{LExp}$$
$$A ::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \forall i.A \in \mathit{Assn}$$

- Thus: $\mathit{AExp} \subsetneq \mathit{LExp}$, $\mathit{BExp} \subsetneq \mathit{Assn}$
- The following (and other) abbreviations will be employed:

$$A_1 \Rightarrow A_2 := \neg A_1 \vee A_2$$
$$\exists i.A := \neg(\forall i.\neg A)$$
$$a_1 \geq a_2 := a_1{>}a_2 \vee a_1{=}a_2$$
$$\vdots$$

Outline of Lecture 9

Semantics of *LExp*

The semantics now additionally depends on values of logical variables:

Definition 9.3 (Semantics of *LExp*)

An interpretation is an element of the set $Int := \{I \mid I : LVar \to \mathbb{Z}\}$. The value of an arithmetic expressions with logical variables is given by the functional

$$\mathfrak{L}[\![.]\!] : LExp \to (Int \to (\Sigma \to \mathbb{Z}))$$

where

$$\begin{array}{ll} \mathfrak{L}[\![z]\!]I\sigma := z & \mathfrak{L}[\![a_1{+}a_2]\!]I\sigma := \mathfrak{L}[\![a_1]\!]I\sigma + \mathfrak{L}[\![a_2]\!]I\sigma \\ \mathfrak{L}[\![x]\!]I\sigma := \sigma(x) & \mathfrak{L}[\![a_1{-}a_2]\!]I\sigma := \mathfrak{L}[\![a_1]\!]I\sigma - \mathfrak{L}[\![a_2]\!]I\sigma \\ \mathfrak{L}[\![i]\!]I\sigma := I(i) & \mathfrak{L}[\![a_1{*}a_2]\!]I\sigma := \mathfrak{L}[\![a_1]\!]I\sigma \cdot \mathfrak{L}[\![a_2]\!]I\sigma \end{array}$$

Semantics of *LExp*

The semantics now additionally depends on values of logical variables:

Definition 9.3 (Semantics of *LExp*)

An interpretation is an element of the set $Int := \{I \mid I : LVar \to \mathbb{Z}\}$. The value of an arithmetic expressions with logical variables is given by the functional

$$\mathfrak{L}[\![.]\!] : LExp \to (Int \to (\Sigma \to \mathbb{Z}))$$

where

$$\begin{aligned} \mathfrak{L}[\![z]\!]I\sigma &:= z & \mathfrak{L}[\![a_1{+}a_2]\!]I\sigma &:= \mathfrak{L}[\![a_1]\!]I\sigma + \mathfrak{L}[\![a_2]\!]I\sigma \\ \mathfrak{L}[\![x]\!]I\sigma &:= \sigma(x) & \mathfrak{L}[\![a_1{-}a_2]\!]I\sigma &:= \mathfrak{L}[\![a_1]\!]I\sigma - \mathfrak{L}[\![a_2]\!]I\sigma \\ \mathfrak{L}[\![i]\!]I\sigma &:= I(i) & \mathfrak{L}[\![a_1{*}a_2]\!]I\sigma &:= \mathfrak{L}[\![a_1]\!]I\sigma \cdot \mathfrak{L}[\![a_2]\!]I\sigma \end{aligned}$$

Definition 6.1 (denotational semantics of arithmetic expressions) implies:

Corollary 9.4

For every $a \in AExp$ (without logical variables), $I \in Int$, and $\sigma \in \Sigma$:

$$\mathfrak{L}[\![a]\!]I\sigma = \mathfrak{A}[\![a]\!]\sigma.$$

Semantics of Assertions I

- Formalized by a satisfaction relation of the form

$$\sigma \models A$$

(where $\sigma \in \Sigma$ and $A \in \mathit{Assn}$)

Semantics of Assertions I

- Formalized by a satisfaction relation of the form

$$\sigma \models A$$

  (where $\sigma \in \Sigma$ and $A \in \mathit{Assn}$)
- Non-terminating computations captured by undefined state $\perp$

Semantics of Assertions I

- Formalized by a satisfaction relation of the form

$$\sigma \models A$$

  (where $\sigma \in \Sigma$ and $A \in \mathit{Assn}$)
- Non-terminating computations captured by undefined state $\perp$
- Modification of interpretations (in analogy to program states):

$$I[i \mapsto z](j) := \begin{cases} z & \text{if } j = i \\ I(j) & \text{otherwise} \end{cases}$$

Semantics of Assertions II

Reminder: $A ::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \forall i.A \in Assn$

Definition 9.5 (Semantics of assertions)

Let $A \in Assn$, $\sigma \in \Sigma$, and $I \in Int$. The relation "σ satisfies A in I" (notation: $\sigma \models^I A$) is inductively defined by:

$$
\begin{array}{ll}
\sigma \models^I \text{true} & \\
\sigma \models^I a_1{=}a_2 & \text{if } \mathfrak{L}[\![a_1]\!] I\sigma = \mathfrak{L}[\![a_2]\!] I\sigma \\
\sigma \models^I a_1{>}a_2 & \text{if } \mathfrak{L}[\![a_1]\!] I\sigma > \mathfrak{L}[\![a_2]\!] I\sigma \\
\sigma \models^I \neg A & \text{if not } \sigma \models^I A \\
\sigma \models^I A_1 \wedge A_2 & \text{if } \sigma \models^I A_1 \text{ and } \sigma \models^I A_2 \\
\sigma \models^I A_1 \vee A_2 & \text{if } \sigma \models^I A_1 \text{ or } \sigma \models^I A_2 \\
\sigma \models^I \forall i.A & \text{if } \sigma \models^{I[i \mapsto z]} A \text{ for every } z \in \mathbb{Z}
\end{array}
$$

Furthermore σ satisfies A ($\sigma \models A$) if $\sigma \models^I A$ for every interpretation $I \in Int$, and A is called valid ($\models A$) if $\sigma \models A$ for every state $\sigma \in \Sigma$.

Semantics of Assertions III

Example 9.6

The following assertion expresses that, in the current state $\sigma \in \Sigma$, $\sigma(y)$ is the greatest divisor of $\sigma(x)$:

$$(\exists i.i > 1 \wedge i{*}y = x) \wedge \forall j.\forall k.(j > 1 \wedge j{*}k = x \Rightarrow k \leq y)$$

Semantics of Assertions III

Example 9.6

The following assertion expresses that, in the current state $\sigma \in \Sigma$, $\sigma(y)$ is the greatest divisor of $\sigma(x)$:

$$(\exists i.i > 1 \wedge i{*}y = x) \wedge \forall j.\forall k.(j > 1 \wedge j{*}k = x \Rightarrow k \leq y)$$

In analogy to Corollary 9.4, Definition 6.2 (denotational semantics of Boolean expressions) yields:

Corollary 9.7

For every $b \in BExp$ (without logical variables), $I \in Int$, and $\sigma \in \Sigma$:

$$\sigma \models^I b \iff \mathfrak{B}[\![b]\!]\sigma = \text{true}.$$

Semantics of Assertions IV

Definition 9.8 (Extension)

Let $A \in \mathit{Assn}$ and $I \in \mathit{Int}$. The extension of A with respect to I is given by

$$A^I := \{\sigma \in \Sigma \mid \sigma \models^I A\}.$$

Semantics of Assertions IV

Definition 9.8 (Extension)

Let $A \in \mathit{Assn}$ and $I \in \mathit{Int}$. The extension of A with respect to I is given by

$$A^I := \{\sigma \in \Sigma \mid \sigma \models^I A\}.$$

Example 9.9

For $A := (\exists i.\, i*i = x)$ and every $I \in \mathit{Int}$,

$$A^I = \{\sigma \in \Sigma \mid \sigma(x) \in \{0, 1, 4, 9, \ldots\}\}$$

Outline of Lecture 9

Partial Correctness Properties

Definition 9.10 (Partial correctness properties)

Let $A, B \in Assn$ and $c \in Cmd$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property (PCP) with precondition A and postcondition B.

Partial Correctness Properties

Definition 9.10 (Partial correctness properties)

Let $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property (PCP) with precondition A and postcondition B.
- Given $\sigma \in \Sigma$ and $I \in \mathit{Int}$, we let

$$\sigma \models^I \{A\}\, c\, \{B\}$$

if ($\sigma \models^I A$ and $\mathfrak{C}[\![c]\!]\sigma \neq \bot$) implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
(or equivalently: $\sigma \in A^I \Rightarrow \mathfrak{C}[\![c]\!]\sigma \in B^I \cup \{\bot\}$).

Partial Correctness Properties

Definition 9.10 (Partial correctness properties)

Let $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property (PCP) with precondition A and postcondition B.
- Given $\sigma \in \Sigma$ and $I \in \mathit{Int}$, we let

$$\sigma \models^I \{A\}\, c\, \{B\}$$

  if ($\sigma \models^I A$ and $\mathfrak{C}[\![c]\!]\sigma \neq \bot$) implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
  (or equivalently: $\sigma \in A^I \Rightarrow \mathfrak{C}[\![c]\!]\sigma \in B^I \cup \{\bot\}$).
- $\{A\}\, c\, \{B\}$ is called valid in I (notation: $\models^I \{A\}\, c\, \{B\}$) if $\sigma \models^I \{A\}\, c\, \{B\}$ for every $\sigma \in \Sigma$
  (or equivalently: $\mathfrak{C}[\![c]\!]A^I \subseteq B^I \cup \{\bot\}$).

Partial Correctness Properties

Definition 9.10 (Partial correctness properties)

Let $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property (PCP) with precondition A and postcondition B.
- Given $\sigma \in \Sigma$ and $I \in \mathit{Int}$, we let

$$\sigma \models^I \{A\}\, c\, \{B\}$$

  if ($\sigma \models^I A$ and $\mathfrak{C}[\![c]\!]\sigma \neq \bot$) implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
  (or equivalently: $\sigma \in A^I \Rightarrow \mathfrak{C}[\![c]\!]\sigma \in B^I \cup \{\bot\}$).
- $\{A\}\, c\, \{B\}$ is called valid in I (notation: $\models^I \{A\}\, c\, \{B\}$) if $\sigma \models^I \{A\}\, c\, \{B\}$ for every $\sigma \in \Sigma$
  (or equivalently: $\mathfrak{C}[\![c]\!]A^I \subseteq B^I \cup \{\bot\}$).
- $\{A\}\, c\, \{B\}$ is called valid (notation: $\models \{A\}\, c\, \{B\}$) if $\models^I \{A\}\, c\, \{B\}$ for every $I \in \mathit{Int}$.

Outline of Lecture 9

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\, \mathtt{x := x+1}\, \{i < \mathtt{x}\}$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\, \mathtt{x} := \mathtt{x+1}\, \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\, \mathtt{x} := \mathtt{x+1}\, \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\sigma \models^{I} (i \leq \mathtt{x})$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\, \mathtt{x} := \mathtt{x+1}\, \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\, \mathtt{x} := \mathtt{x+1}\, \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\begin{aligned} & \sigma \models^{I} (i \leq \mathtt{x}) \\ \Rightarrow\ & \mathfrak{L}[\![i]\!] I\sigma \leq \mathfrak{L}[\![\mathtt{x}]\!] I\sigma \qquad \text{(Definition 9.5)} \end{aligned}$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\, \mathtt{x := x+1}\, \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\, \mathtt{x := x+1}\, \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\begin{aligned} & \sigma \models^{I} (i \leq \mathtt{x}) \\ \Rightarrow\ & \mathfrak{L}[\![i]\!] I\sigma \leq \mathfrak{L}[\![\mathtt{x}]\!] I\sigma && \text{(Definition 9.5)} \\ \Rightarrow\ & I(i) \leq \sigma(\mathtt{x}) && \text{(Definition 9.3)} \end{aligned}$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\begin{aligned}
& \sigma \models^{I} (i \leq \mathtt{x}) \\
\Rightarrow\ & \mathfrak{L}[\![i]\!] I\sigma \leq \mathfrak{L}[\![\mathtt{x}]\!] I\sigma && \text{(Definition 9.5)} \\
\Rightarrow\ & I(i) \leq \sigma(\mathtt{x}) && \text{(Definition 9.3)} \\
\Rightarrow\ & I(i) < \sigma(\mathtt{x}) + 1 \\
& = (\mathfrak{C}[\![\mathtt{x := x+1}]\!]\sigma)(\mathtt{x})
\end{aligned}$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\begin{aligned}
& \sigma \models^{I} (i \leq \mathtt{x}) \\
\Rightarrow\ & \mathfrak{L}[\![i]\!] I\sigma \leq \mathfrak{L}[\![\mathtt{x}]\!] I\sigma && \text{(Definition 9.5)} \\
\Rightarrow\ & I(i) \leq \sigma(\mathtt{x}) && \text{(Definition 9.3)} \\
\Rightarrow\ & I(i) < \sigma(\mathtt{x}) + 1 \\
& \quad = (\mathfrak{C}[\![\mathtt{x := x+1}]\!]\sigma)(\mathtt{x}) \\
\Rightarrow\ & \mathfrak{C}[\![\mathtt{x := x+1}]\!]\sigma \models^{I} (i < \mathtt{x})
\end{aligned}$$

A Valid Partial Correctness Property

Example 9.11

- Let $\mathtt{x} \in \mathit{Var}$ and $i \in \mathit{LVar}$. We have to show:

$$\models \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

- According to Definition 9.10, this is equivalent to

$$\sigma \models^{I} \{i \leq \mathtt{x}\}\ \mathtt{x := x+1}\ \{i < \mathtt{x}\}$$

for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$, which is entailed by the following implications:

$$\begin{aligned}
& \sigma \models^{I} (i \leq \mathtt{x}) \\
\Rightarrow\ & \mathfrak{L}[\![i]\!] I\sigma \leq \mathfrak{L}[\![\mathtt{x}]\!] I\sigma && \text{(Definition 9.5)} \\
\Rightarrow\ & I(i) \leq \sigma(\mathtt{x}) && \text{(Definition 9.3)} \\
\Rightarrow\ & I(i) < \sigma(\mathtt{x}) + 1 \\
& \quad = (\mathfrak{C}[\![\mathtt{x := x+1}]\!]\sigma)(\mathtt{x}) \\
\Rightarrow\ & \mathfrak{C}[\![\mathtt{x := x+1}]\!]\sigma \models^{I} (i < \mathtt{x}) \\
\Rightarrow\ & \text{claim}
\end{aligned}$$

Outline of Lecture 9

Proof Rules for Partial Correctness

Hoare Logic

Goal: syntactic derivation of valid partial correctness properties.
Here $A[x \mapsto a]$ denotes the syntactic replacement of every occurrence of x by a in A.

Tony Hoare (* 1934)

Definition 9.12 (Hoare Logic)

The Hoare rules are given by

$$\text{(skip)}\ \frac{}{\{A\}\ \texttt{skip}\ \{A\}} \qquad \text{(asgn)}\ \frac{}{\{A[x \mapsto a]\}\ x := a\ \{A\}}$$

$$\text{(seq)}\ \frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1 ; c_2\ \{B\}} \qquad \text{(if)}\ \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}\ \{B\}}$$

$$\text{(while)}\ \frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}\ \{A \wedge \neg b\}} \qquad \text{(cons)}\ \frac{\models (A \Rightarrow A') \quad \{A'\}\ c\ \{B'\} \quad \models (B' \Rightarrow B)}{\{A\}\ c\ \{B\}}$$

A partial correctness property is provable (notation: $\vdash \{A\}\ c\ \{B\}$) if it is derivable by the Hoare rules. In (while), A is called a (loop) invariant.