Semantics and Verification of Software

Summer Semester 2019

Lecture 8: Denotational Semantics of WHILE III (Fixpoint & Coincidence Theorem)

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Outline of Lecture 8

Recap: CCPOs and Continuous Functions

Characterisation of $\mathrm{fix}(\Phi)$ II

Goals:

- Prove existence of $\mathrm{fix}(\Phi)$ for $\Phi(f) = \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$
- Show how it can be "computed" (more exactly: approximated)

Sufficient conditions:

on domain $\Sigma \dashrightarrow \Sigma$: chain-complete partial order

on function Φ: monotonicity and continuity

Chains and Least Upper Bounds I

Definition (Chain, (least) upper bound)

Let $(D, \sqsubseteq)$ be a partial order and $S \subseteq D$.

1. S is called a chain in D if, for every $s_1, s_2 \in S$,

$$s_1 \sqsubseteq s_2 \text{ or } s_2 \sqsubseteq s_1$$

   (that is, S is a totally ordered subset of D).
2. An element $d \in D$ is called an upper bound of S if $s \sqsubseteq d$ for every $s \in S$ (notation: $S \sqsubseteq d$).
3. An upper bound d of S is called least upper bound (LUB) or supremum of S if $d \sqsubseteq d'$ for every upper bound d' of S (notation: $d = \bigsqcup S$).

Chain Completeness

Definition (Chain completeness)

A partial order is called chain complete (CCPO) if each of its chains has a least upper bound.

Recap: CCPOs and Continuous Functions

Monotonicity

Definition (Monotonicity)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders, and let $F : D \to D'$. F is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \Rightarrow F(d_1) \sqsubseteq' F(d_2).$$

Interpretation: monotonic functions “preserve information”

Application to $\mathrm{fix}(\Phi)$

Lemma

Let $b \in BExp$, $c \in Cmd$, and $\Phi : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)$ with $\Phi(f) := \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$. Then Φ is monotonic w.r.t. $(\Sigma \dashrightarrow \Sigma, \sqsubseteq)$.

Proof.

to be shown explicitly □

Monotonicity II

The following lemma states how chains behave under monotonic functions.

Lemma

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be CCPOs, $F : D \to D'$ monotonic, and $S \subseteq D$ a chain in D. Then:

1. $F(S) := \{F(d) \mid d \in S\}$ *is a chain in* D'.
2. $\bigsqcup F(S) \sqsubseteq' F(\bigsqcup S)$.

Proof.

on the board □

Recap: CCPOs and Continuous Functions

Continuity

A function F is continuous if applying F and taking suprema is commutable:

Definition (Continuity)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be CCPOs and $F : D \to D'$ monotonic. Then F is called continuous (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every non-empty chain $S \subseteq D$,

$$F\left(\bigsqcup S\right) = \bigsqcup F(S).$$

Remark: according to Lemma 7.14(1), the monotonicity of F guarantees the existence of $\bigsqcup F(S)$.

Application to $\mathrm{fix}(\Phi)$

Lemma

Let $b \in BExp$, $c \in Cmd$, and $\Phi(f) := \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$. Then Φ is continuous w.r.t. $(\Sigma \dashrightarrow \Sigma, \sqsubseteq)$.

Proof.

Let $S \subseteq \Sigma \dashrightarrow \Sigma$ be a non-empty chain. We have to show that $\Phi\left(\bigsqcup S\right) = \bigsqcup \Phi(S)$.

"$\bigsqcup \Phi(S) \sqsubseteq \Phi\left(\bigsqcup S\right)$": follows from Lemma 7.13 and 7.14(2)

"$\Phi\left(\bigsqcup S\right) \sqsubseteq \bigsqcup \Phi(S)$": on the board

□

Outline of Lecture 8

The Fixpoint Theorem

Alfred Tarski (1901–1983)

Bronislaw Knaster (1893–1990)

Theorem 8.1 (Fixpoint Theorem by Tarski and Knaster)

Let $(D, \sqsubseteq)$ *be a CCPO and* $F : D \to D$ *continuous. Then*

$$\mathrm{fix}(F) := \bigsqcup \left\{ F^n \left(\bigsqcup \emptyset \right) \mid n \in \mathbb{N} \right\}$$

is the least fixpoint of F *where* $F^0(d) := d$ *and* $F^{n+1}(d) := F(F^n(d))$.

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on the board □

An Example

Example 8.2

- **Domain:** $(2^{\mathbb{N}}, \subseteq)$ (CCPO with $\bigsqcup S = \bigcup_{N \in S} N$ – see Example 7.7)

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  - F monotonic: $M \subseteq N \Rightarrow F(M) = M \cup A \subseteq N \cup A = F(N)$
  - F continuous: $F(\bigsqcup S) = F\left(\bigcup_{N \in S} N\right) = \left(\bigcup_{N \in S} N\right) \cup A = \bigcup_{N \in S} (N \cup A) = \bigcup_{N \in S} F(N) = \bigsqcup F(S)$

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- **Fixpoint iteration:** $N_n := F^n(\bigsqcup \emptyset)$ where $\bigsqcup \emptyset = \emptyset$
  - $N_0 = \bigsqcup \emptyset = \emptyset$
  - $N_1 = F(N_0) = \emptyset \cup A = A$
  - $N_2 = F(N_1) = A \cup A = A = N_n$ for every $n \geq 1$

$\Rightarrow \mathrm{fix}(F) = A$ (least $N \subseteq \mathbb{N}$ such that $N \cup A = N$)

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  $\Rightarrow \mathrm{fix}(F) = A$ (least $N \subseteq \mathbb{N}$ such that $N \cup A = N$)
- **Alternatively:** $F(N) := N \cap A$

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The Fixpoint Theorem

An Example

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- **Function:** $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : N \mapsto N \cup A$ for some fixed $A \subseteq \mathbb{N}$
  - F monotonic: $M \subseteq N \Rightarrow F(M) = M \cup A \subseteq N \cup A = F(N)$
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- **Fixpoint iteration:** $N_n := F^n(\bigsqcup \emptyset)$ where $\bigsqcup \emptyset = \emptyset$
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Remark: in general, the fixpoint is only reached in the limit (cf. Example 8.4)

Outline of Lecture 8

Application to $\mathrm{fix}(\Phi)$

Altogether this completes the definition of $\mathfrak{C}[\![.]\!]$. In particular, for the `while` statement:

Corollary 8.3

Let $b \in BExp$, $c \in Cmd$, and $\Phi(f) := \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$. Then

$$\mathrm{graph}(\mathrm{fix}(\Phi)) = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\Phi^n(f_\emptyset))$$

Application to $\mathrm{fix}(\Phi)$

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Proof.

Using

- Lemma 7.9
  - $(\Sigma \dashrightarrow \Sigma, \sqsubseteq)$ CCPO with least element $f_\emptyset$
  - LUB = union of graphs
- Lemma 7.16 (Φ continuous)
- Theorem 8.1 (Fixpoint Theorem)

□

Denotational Semantics of Factorial Program I

Example 8.4 (Factorial program)

- Let $c \in \mathit{Cmd}$ be given by `y:=1; while ¬(x=1) do y:=y*x; x:=x-1 end`

Denotational Semantics of Factorial Program I

Example 8.4 (Factorial program)

- Let $c \in \mathit{Cmd}$ be given by `y:=1; while ¬(x=1) do y:=y*x; x:=x-1 end`
- For every initial state $\sigma_0 \in \Sigma$, Definition 6.3 yields:

$$\mathfrak{C}[\![c]\!](\sigma_0) = \mathrm{fix}(\Phi)(\sigma_1)$$

where $\sigma_1 := \sigma_0[\mathtt{y} \mapsto 1]$ and, for every $f : \Sigma \dashrightarrow \Sigma$ and $\sigma \in \Sigma$,

$$\begin{aligned}\Phi(f)(\sigma) &= \mathrm{cond}(\mathfrak{B}[\![\neg(\mathtt{x}=1)]\!], f \circ \mathfrak{C}[\![\mathtt{y:=y*x;\ x:=x-1}]\!], \mathrm{id}_\Sigma)(\sigma) \\ &= \begin{cases} \sigma & \text{if } \sigma(\mathtt{x}) = 1 \\ f(\sigma') & \text{otherwise} \end{cases}\end{aligned}$$

with $\sigma' := \sigma[\mathtt{y} \mapsto \sigma(\mathtt{y}) * \sigma(\mathtt{x}), \mathtt{x} \mapsto \sigma(\mathtt{x}) - 1]$.

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with $\sigma' := \sigma[\mathtt{y} \mapsto \sigma(\mathtt{y}) * \sigma(\mathtt{x}), \mathtt{x} \mapsto \sigma(\mathtt{x}) - 1]$.

- Approximations of least fixpoint of Φ according to Theorem 8.1:

$$\mathrm{fix}(\Phi) = \bigsqcup \{\Phi^n(f_\emptyset) \mid n \in \mathbb{N}\}$$

(where $\mathrm{graph}(f_\emptyset) = \emptyset$)

Denotational Semantics of Factorial Program II

Reminder: $\Phi(f)(\sigma) = \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f(\sigma') & \text{otherwise} \end{cases} \qquad \sigma' = \sigma[\mathrm{y} \mapsto \sigma(\mathrm{y}) * \sigma(\mathrm{x}), \mathrm{x} \mapsto \sigma(\mathrm{x}) - 1]$

Example 8.4 (Factorial program; continued)

$$\begin{aligned} f_0(\sigma) &:= \Phi^0(f_\emptyset)(\sigma) \\ &= f_\emptyset(\sigma) \\ &= \text{undefined} \end{aligned}$$

Denotational Semantics of Factorial Program II

Reminder: $\Phi(f)(\sigma) = \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f(\sigma') & \text{otherwise} \end{cases} \qquad \sigma' = \sigma[\mathrm{y} \mapsto \sigma(\mathrm{y}) * \sigma(\mathrm{x}), \mathrm{x} \mapsto \sigma(\mathrm{x}) - 1]$

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&= \Phi(f_0)(\sigma) \\
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\end{aligned}
$$

Denotational Semantics of Factorial Program II

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&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \text{undefined} & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_2(\sigma) &:= \Phi^2(f_\emptyset)(\sigma) \\
&= \Phi(f_1)(\sigma) \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f_1(\sigma') & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \sigma' & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) = 1 \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) \neq 1 \end{cases} \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \sigma' & \text{if } \sigma(\mathrm{x}) = 2 \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma(\mathrm{x}) \neq 2 \end{cases} \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \sigma[\mathrm{y} \mapsto 2 * \sigma(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } \sigma(\mathrm{x}) = 2 \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma(\mathrm{x}) \neq 2 \end{cases}
\end{aligned}
$$

Denotational Semantics of Factorial Program III

Reminder: $\Phi(f)(\sigma) = \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f(\sigma') & \text{otherwise} \end{cases} \qquad \sigma' = \sigma[\mathrm{y} \mapsto \sigma(\mathrm{y}) * \sigma(\mathrm{x}), \mathrm{x} \mapsto \sigma(\mathrm{x}) - 1]$

Example 8.4 (Factorial program; continued)

$$
\begin{aligned}
f_3(\sigma) &:= \Phi^3(f_\emptyset)(\sigma) \\
&= \Phi(f_2)(\sigma) \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f_2(\sigma') & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \sigma' & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) = 1 \\ \sigma'[\mathrm{y} \mapsto 2 * \sigma'(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) = 2 \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) \neq 1, \sigma'(\mathrm{x}) \neq 2 \end{cases} \\
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&= \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ \sigma[\mathrm{y} \mapsto 2 * \sigma(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } \sigma(\mathrm{x}) = 2 \\ \sigma[\mathrm{y} \mapsto 3 * 2 * \sigma(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } \sigma(\mathrm{x}) = 3 \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \notin \{1, 2, 3\} \end{cases}
\end{aligned}
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Denotational Semantics of Factorial Program IV

Reminder: $\Phi(f)(\sigma) = \begin{cases} \sigma & \text{if } \sigma(\mathrm{x}) = 1 \\ f(\sigma') & \text{otherwise} \end{cases} \qquad \sigma' = \sigma[\mathrm{y} \mapsto \sigma(\mathrm{y}) * \sigma(\mathrm{x}), \mathrm{x} \mapsto \sigma(\mathrm{x}) - 1]$

Example 8.4 (Factorial program; continued)

- n-th approximation:

$$
\begin{aligned}
& f_n(\sigma) \\
& := \Phi^n(f_\emptyset)(\sigma) \\
& = \begin{cases} \sigma[\mathrm{y} \mapsto \sigma(\mathrm{x}) * (\sigma(\mathrm{x}) - 1) * \ldots * 2 * \sigma(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } 1 \leq \sigma(\mathrm{x}) \leq n \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \notin \{1, \ldots, n\} \end{cases} \\
& = \begin{cases} \sigma[\mathrm{y} \mapsto (\sigma(\mathrm{x}))! * \sigma(\mathrm{y}), \mathrm{x} \mapsto 1] & \text{if } 1 \leq \sigma(\mathrm{x}) \leq n \\ \text{undefined} & \text{if } \sigma(\mathrm{x}) \notin \{1, \ldots, n\} \end{cases}
\end{aligned}
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Denotational Semantics of Factorial Program IV

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Example 8.4 (Factorial program; continued)

- n-th approximation:

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- Fixpoint:

$$\mathfrak{C}[\![c]\!](\sigma_0) = \mathrm{fix}(\Phi)(\sigma_1) = \begin{cases} \sigma[\mathrm{y} \mapsto (\sigma(\mathrm{x}))!, \mathrm{x} \mapsto 1] & \text{if } \sigma(\mathrm{x}) \geq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

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Summary: Denotational Semantics

- Semantic model: partial state transformations ($\Sigma \dashrightarrow \Sigma$)

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- Capturing the recursive nature of loops by a fixpoint definition (for a continuous function on a CCPO)

Summary: Denotational Semantics

- Semantic model: partial state transformations ($\Sigma \dashrightarrow \Sigma$)
- Compositional definition of functional $\mathfrak{C}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$
- Capturing the recursive nature of loops by a fixpoint definition (for a continuous function on a CCPO)
- Approximation by fixpoint iteration

Outline of Lecture 8

Equivalence of Semantics I

Remember: in Definition 4.1, $\mathfrak{D}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$ was given by

$$\mathfrak{D}[\![c]\!](\sigma) = \sigma' \iff \langle c, \sigma \rangle \to \sigma'$$

Equivalence of Semantics I

Remember: in Definition 4.1, $\mathfrak{O}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$ was given by

$$\mathfrak{O}[\![c]\!](\sigma) = \sigma' \iff \langle c, \sigma \rangle \to \sigma'$$

Theorem 8.5 (Coincidence Theorem)

For every $c \in \mathit{Cmd}$,

$$\mathfrak{O}[\![c]\!] = \mathfrak{C}[\![c]\!],$$

i.e., $\langle c, \sigma \rangle \to \sigma'$ *iff* $\mathfrak{C}[\![c]\!](\sigma) = \sigma'$, *and thus* $\mathfrak{O}[\![.]\!] = \mathfrak{C}[\![.]\!]$.

Equivalence of Semantics II

The proof of Theorem 8.5 employs the following auxiliary propositions:

Lemma 8.6

1. *For every* $a \in AExp$, $\sigma \in \Sigma$, *and* $z \in \mathbb{Z}$*:*

$$\langle a, \sigma \rangle \to z \iff \mathfrak{A}[\![a]\!](\sigma) = z.$$

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1. *For every* $a \in AExp$, $\sigma \in \Sigma$, *and* $z \in \mathbb{Z}$:

$$\langle a, \sigma \rangle \to z \iff \mathfrak{A}[\![a]\!](\sigma) = z.$$

2. *For every* $b \in BExp$, $\sigma \in \Sigma$, *and* $t \in \mathbb{B}$:

$$\langle b, \sigma \rangle \to t \iff \mathfrak{B}[\![b]\!](\sigma) = t.$$

Equivalence of Semantics II

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1. *For every* $a \in AExp$, $\sigma \in \Sigma$, *and* $z \in \mathbb{Z}$*:*

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2. *For every* $b \in BExp$, $\sigma \in \Sigma$, *and* $t \in \mathbb{B}$*:*

$$\langle b, \sigma \rangle \to t \iff \mathfrak{B}[\![b]\!](\sigma) = t.$$

Proof.

1. structural induction on a
2. structural induction on b

□

Equivalence of Semantics III

Proof (Theorem 8.5).

We have to show that

$$\langle c, \sigma \rangle \to \sigma' \iff \mathfrak{C}[\![c]\!](\sigma) = \sigma'$$

- $\Rightarrow$ by structural induction over the derivation tree of $\langle c, \sigma \rangle \to \sigma'$
- $\Leftarrow$ by structural induction over c (with a nested complete induction over fixpoint index n)

(on the board) □

Overview: Operational/Denotational Semantics

Definition (3.2; Execution relation for statements)

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, \sigma\rangle \to \sigma} \qquad \text{(asgn)}\ \frac{\langle a, \sigma\rangle \to z}{\langle x := a, \sigma\rangle \to \sigma[x \mapsto z]}$$

$$\text{(seq)}\ \frac{\langle c_1, \sigma\rangle \to \sigma' \quad \langle c_2, \sigma'\rangle \to \sigma''}{\langle c_1 ; c_2, \sigma\rangle \to \sigma''} \qquad \text{(if-t)}\ \frac{\langle b, \sigma\rangle \to \text{true} \quad \langle c_1, \sigma\rangle \to \sigma'}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end}, \sigma\rangle \to \sigma'}$$

$$\text{(if-f)}\ \frac{\langle b, \sigma\rangle \to \text{false} \quad \langle c_2, \sigma\rangle \to \sigma'}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end}, \sigma\rangle \to \sigma'} \qquad \text{(wh-f)}\ \frac{\langle b, \sigma\rangle \to \text{false}}{\langle \texttt{while } b \texttt{ do } c \texttt{ end}, \sigma\rangle \to \sigma}$$

$$\text{(wh-t)}\ \frac{\langle b, \sigma\rangle \to \text{true} \quad \langle c, \sigma\rangle \to \sigma' \quad \langle \texttt{while } b \texttt{ do } c \texttt{ end}, \sigma'\rangle \to \sigma''}{\langle \texttt{while } b \texttt{ do } c \texttt{ end}, \sigma\rangle \to \sigma''}$$

Definition (6.3; Denotational semantics of statements)

$$\begin{aligned}
\mathfrak{C}[\![\texttt{skip}]\!] &:= \mathrm{id}_\Sigma \\
\mathfrak{C}[\![x := a]\!]\sigma &:= \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \\
\mathfrak{C}[\![c_1 ; c_2]\!] &:= \mathfrak{C}[\![c_2]\!] \circ \mathfrak{C}[\![c_1]\!] \\
\mathfrak{C}[\![\texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end}]\!] &:= \mathrm{cond}(\mathfrak{B}[\![b]\!], \mathfrak{C}[\![c_1]\!], \mathfrak{C}[\![c_2]\!]) \\
\mathfrak{C}[\![\texttt{while } b \texttt{ do } c \texttt{ end}]\!] &:= \mathrm{fix}(\Phi) \text{ where } \Phi(f) := \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)
\end{aligned}$$