Exercise Sheet 1

Task 1

a) $z[x := a'] := z$

$$y[x:=a'] := \begin{cases} a', & \text{if } y = x \\ y, & \text{if } y \neq x \end{cases}$$

$$(a_1 \circ a_2)[x:=a'] := a_1[x:=a'] \circ a_2[x:=a']$$

for $\circ \in \{-, +, \cdot\}$

"Iverson bracket"

b) $occ(z, x) := 0$ $\qquad occ(y, x) := [x = y] = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$

$$occ(a_1 \circ a_2, x) := occ(a_1, x) + occ(a_2, x)$$

c) Induction base:

$\underline{a = z}$

$$FV(z[x:=a']) = FV(z) = \emptyset \subseteq (FV(z) \setminus \{x\}) \cup FV(a')$$

$\underline{a = x}$

$$FV(x[x:=a']) = FV(a') \subseteq (FV(x) \setminus \{x\}) \cup FV(a')$$

$\underline{a = y \neq x}$

$$FV(y[x:=a']) = FV(y) = FV(y) \setminus \{x\} \subseteq (FV(y) \setminus \{x\}) \cup FV(a')$$

Induction step:

$a = a_1 \circ a_2$, $\circ \in \{-, +, \cdot\}$

$$FV((a_1 \circ a_2)[x := a'])$$

$$= FV(a_1[x:=a'] \circ a_2[x:=a'])$$

$$= FV(a_1[x:=a']) \cup FV(a_2[x:=a'])$$

$$\overset{I.H.}{\subseteq} (FV(a_1) \setminus \{x\} \cup FV(a')) \cup (FV(a_2) \setminus \{x\} \cup FV(a'))$$

$$= ((FV(a_1) \cup FV(a_2)) \setminus \{x\}) \cup FV(a')$$

$$= (FV(a_1 \circ a_1) \setminus \{x\}) \cup FV(a')$$

d) (i) $length(a[x:=a']) = length(a) + occ(a,x) \cdot (length(a') - 1)$

(ii) By induction on the structure of a.

Induction base:

For $y \neq x$ or y constant:

$$length(y[x:=a']) = length(y) = length(y) + 0 \cdot (length(a') - 1)$$

$$= length(y) + occ(y,x) \cdot (length(a') - 1)$$

$a = x$

$$length(x[x:=a']) = length(a') = 1 + (length(a') - 1)$$

$$= length(x) + occ(x,x) \cdot (length(a') - 1)$$

Induction step:

$\underline{a = a_1 \circ a_2}$, $\circ \in \{-, +, \cdot\}$

$$\begin{aligned}
\mathrm{length}((a_1 \circ a_2)[x:=a']) &= \mathrm{length}(a_1[x:=a'] \circ a_2[x:=a']) \\
&= 1 + \mathrm{length}(a_1[x:=a']) + \mathrm{length}(a_2[x:=a']) \\
&\overset{\text{I.H.}}{=} 1 + \mathrm{length}(a_1) + \mathrm{occ}(a_1, x) \cdot (\mathrm{length}(a') - 1) \\
&\quad + \mathrm{length}(a_2) + \mathrm{occ}(a_2, x) \cdot (\mathrm{length}(a') - 1) \\
&= \mathrm{length}(a_1 \circ a_2) + (\mathrm{occ}(a_1, x) + \mathrm{occ}(a_2, x)) \cdot (\mathrm{length}(a') - 1) \\
&= \mathrm{length}(a_1 \circ a_2) + \mathrm{occ}(a_1 \circ a_2, x) \cdot (\mathrm{length}(a') - 1)
\end{aligned}$$

Task 2

By structural induction on the syntax of a

Induction base

$a = z$ There is exactly one derivation tree, $\dfrac{}{\langle z, \sigma \rangle \to z}$.

Hence, $\langle z, \sigma \rangle \to z_1$ and $\langle z, \sigma \rangle \to z_2$ implies $z_1 = z = z_2$.

$a = x$ There is exactly one derivation tree, $\dfrac{}{\langle x, \sigma \rangle \to \sigma(x)}$.

Hence, $\langle x, \sigma \rangle \to z_1$ and $\langle x, \sigma \rangle \to z_2$ implies $z_1 = \sigma(x) = z_2$.

$a = a_1 \circ a_2$ For each $\circ \in \{-, +, \cdot\}$, there is exactly one rule applicable.

Now, if $\langle a_1 \circ a_2, \sigma \rangle \to z_1$ and $\langle a_1 \circ a_2, \sigma \rangle \to z_2$, we get

two derivation trees:

$$\frac{\langle a_1, \sigma \rangle \to z_1' \quad \langle a_2, \sigma \rangle \to z_2'}{\langle a_1 \circ a_2, \sigma \rangle \to z_1}, \text{ where } z_1 := z_1' \circ z_2'$$

$$\frac{\langle a_1, \sigma \rangle \to z_1'' \quad \langle a_2, \sigma \rangle \to z_2''}{\langle a_1 \circ a_2, \sigma \rangle \to z_2}, \text{ where } z_2 := z_1'' \circ z_2''$$

By I.H. $z_1' = z_1''$ and $z_2' = z_2''$. Hence,

$z_1 = z_1' \circ z_2' = z_1'' \circ z_2'' = z_2$.

Task 3

Recall structural induction.

Given: A set S whose elements are either

1) atomic elements x,

2) or composed elements $f(s_1, \dots, s_n)$

for some function $f : S^n \to S$, $n \in \mathbb{N}$.

Now, define $< \subseteq S \times S$ as

$$s < s' \quad \text{iff} \quad s' = f(s_1, \dots, s_n) \text{ and } s = s_i$$

for some $1 \leq i \leq n$.

Clearly, $<$ is a well-ordering as all atomic elements are minimal and every composed element results from finitely many function applications.

Now, consider well-founded induction for $<$ and some proposition P. To show $\forall s.\ P(s)$, we have to show for all $s \in S$ that

$$\forall s' : \left(s' < s \Rightarrow P(s') \right) \Rightarrow P(s).$$

If s is atomic, there is no $s' < s$.

$\underbrace{\text{Hence, we have to show } P(s).}_{\text{Induction Base}}$

If $s = P(s_1, \dots, s_n)$ then $s_1, \dots, s_n < s$.

$\underbrace{\text{Hence, } P(s_1), \dots, P(s_n) \text{ holds.}}_{\text{Induction hypothesis}}$

$\underbrace{\text{We then have to show that } P(f(s_1, \dots, s_n)) \text{ holds}}_{\text{Induction step}}$