Exercise Sheet 3

Due date: May 10th. Please hand in your solutions at the start of the exercise class.

Task 1: Operational Equivalence (30 points)

Recall the $\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b)$ construct from exercise sheet 2. Prove or disprove:

$$\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b) \quad \sim \quad c; \mathsf{while}\ (b)\ \mathsf{do}\ \{c\}\ ,$$

where $\sim$ denotes operational equivalence.

Task 2: Chain Complete Partial Orders (40 points)

Determine whether each of the following statements is true or false. For true statements present a formal proof, and for false statements provide a counterexample.

1. Every continuous function $f\colon (D_1, \sqsubseteq_1) \to (D_1, \sqsubseteq_2)$ between two CCPOs $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ is monotonic.
2. Consider the partial order $(\mathbb{Q}, \leq)$ of the rational numbers ordered by the natural order in the reals. $(\mathbb{Q}, \leq)$ is chain complete.
3. If $f\colon (D_1, \sqsubseteq_1) \to (D_1, \sqsubseteq_2)$ is a monotonic function between two CCPOs and $D \subseteq D_1$ is a chain, then $f(\bigsqcup D) \sqsubseteq_2 \bigsqcup f(D)$.
4. Let $(D, \sqsubseteq)$ be a partial order and let $f\colon (D, \sqsubseteq) \to (D, \sqsubseteq)$ be monotonic. If p is the least element in D satisfying $f(p) \sqsubseteq p$, then p is a fixed point of f.

Task 3: Closed Sets (30 points)

A set $C \subseteq D$ is *closed* if and only if for each chain $G \subseteq C$, $\bigsqcup G \in C$. In the following, let $(D, \sqsubseteq)$ be a chain complete partial order and $f\colon D \to D$ be a continous function. Prove the following two statements:

1. For each closed set $C \subseteq D$ with $f(x) \in C$ for each $x \in C$, we have $\mathsf{fix}(f) \in C$.
2. $f(x) \sqsubseteq x$ implies $\mathsf{fix}(f) \sqsubseteq x$, $x \in D$.