Exercise Sheet 1

Due date: April 26th. Please hand in your solutions at the start of the exercise class.

Task 1: Recursion and Structural Induction (60 points)

Consider the set of arithmetical expressions AExp given by grammar

$$a ::= z \mid x \mid a - a \mid a + a \mid a * a \ .$$

Here z ranges over the set of integers $\mathbb{Z}$ and x over the set of program variables Var.

(a) Give a recursive definition of the *textual substitution* operator $a\,[x := a']$ that replaces every ocurrence of variable x in expression a with expression a'. For example, we have

$$(x + x * y)\,[x := 3 + z] = (3 + z) + (3 + z) * y \ .$$

(b) Give a recursive definition of function $\texttt{occ} : \mathsf{AExp} \times \mathsf{Var} \to \mathbb{N}$ that counts the number of occurrences of a variable within an arithmetic expression. For instance, we should have

$$\texttt{occ}(x + x * y, x) = 2 \ .$$

(c) Show by induction on the structure of a that

$$\texttt{FV}(a\left[x := a'\right]) \subseteq (\texttt{FV}(a) \setminus \{x\}) \cup \texttt{FV}(a') \ .$$

(d) Consider the recursive function $\texttt{length} : \mathsf{AExp} \to \mathbb{N}$ defined by clauses

$$\texttt{length}(z) = \texttt{length}(x) = 1$$
$$\texttt{length}(a_1 \oplus a_2) = 1 + \texttt{length}(a_1) + \texttt{length}(a_2) \quad \text{for } \oplus \in \{-, +, *\} \ .$$

(i) Determine $\texttt{length}(a\,[x := a'])$ in terms of $\texttt{occ}(a, x)$, $\texttt{length}(a)$ and $\texttt{length}(a')$.

(ii) Prove that your proposed formula in (i) is correct.

Task 2: Determinism (20 points)

Prove that Lemma 3.6.1 from the lecture is correct. That is, show that for every $a \in \mathsf{AExp}$, $\sigma \in \Sigma$, and $z, z' \in \mathbb{Z}$, we have

$$\langle a, \sigma \rangle \to z \quad \text{and} \quad \langle a, \sigma \rangle \to z' \qquad \text{implies} \qquad z = z'$$

Task 3: Well-founded Induction (20 points)

A binary relation $< \subseteq S \times S$ is *well-founded* if every non-empty subset $X \subseteq S$ has a minimal element with respect to $<$.

Assume we are given a well-founded relation $< \subseteq S \times S$ and a property P.

Then the principle of *well-founded induction* states the following: In order to show that $P(s)$ holds for all elements $s \in S$, it suffices to prove for all $s \in S$ that $P(s)$ holds under the assumption that $P(s')$ holds for all $s' < s$.

Show that every proof by structural induction can also be written as a proof by well-founded induction.