Static Program Analysis

Lecture 2: Dataflow Analysis I
(Introduction & Available Expressions/Live Variables Analysis)

Summer Semester 2018

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https://moves.rwth-aachen.de/teaching/ss-18/spa/

Exercises

- Assignments made available via L2P Learning Room
- Requires registration via Campus web page
- Please work on assignments in groups of three

Outline of Lecture 2

Preliminaries on Dataflow Analysis

An Example: Available Expressions Analysis

Another Example: Live Variables Analysis

Heading for a Dataflow Analysis Framework

Dataflow Analysis: the Approach

- Traditional form of program analysis

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- Idea: describe how analysis information flows through program

Dataflow Analysis: the Approach

- Traditional form of program analysis
- Idea: describe how analysis information flows through program
- Distinctions:

dependence on statement order:

flow-sensitive vs. flow-insensitive analyses

direction of flow:

forward vs. backward analyses

quantification over paths:

may (union) vs. must (intersection) analyses

procedures:

interprocedural vs. intraprocedural analyses

Labelled Programs

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  - `skip` statements
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Definition 2.1 (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$\begin{aligned}
a &::= z \mid x \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in AExp\\
b &::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg b \mid b_1{\wedge}b_2 \mid b_1{\vee}b_2 \in BExp\\
c &::= [\texttt{skip}]^l \mid [x := a]^l \mid c_1 ; c_2 \mid\\
&\quad\;\; \texttt{if}\ [b]^l\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end} \mid \texttt{while}\ [b]^l\ \texttt{do}\ c\ \texttt{end} \in Cmd
\end{aligned}$$

- All labels in $c \in Cmd$ assumed distinct, denoted by Lab_c
- Labelled fragments of c called blocks, denoted by Blk_c

A WHILE Program

Example 2.2

```
x := 6;
y := 7;
z := 0;
while x > 0 do
  x := x - 1;
  v := y;
  while v > 0 do
    v := v - 1;
    z := z + 1
  end
end
```

A WHILE Program with Labels

Example 2.2

```
[x := 6]^1;
[y := 7]^2;
[z := 0]^3;
while [x > 0]^4 do
  [x := x - 1]^5;
  [v := y]^6;
  while [v > 0]^7 do
    [v := v - 1]^8;
    [z := z + 1]^9
  end
end
```

Representing Control Flow I

Every (labelled) statement has a single entry (given by the initial label) and generally multiple exits (given by the final labels):

Definition 2.3 (Initial and final labels)

The mappings

$$\mathsf{init} : \mathit{Cmd} \to \mathit{Lab} \quad \text{and} \quad \mathsf{final} : \mathit{Cmd} \to 2^{\mathit{Lab}}$$

respectively return the initial and final label(s) of a statement:

$c \in \mathit{Cmd}$	$\mathsf{init}(c)$	$\mathsf{final}(c)$
$[\mathtt{skip}]^l$	l	$\{l\}$
$[x := a]^l$	l	$\{l\}$
$c_1 ; c_2$	$\mathsf{init}(c_1)$	$\mathsf{final}(c_2)$
$\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \mathtt{end}$	l	$\mathsf{final}(c_1) \cup \mathsf{final}(c_2)$
$\mathtt{while}\ [b]^l\ \mathtt{do}\ c\ \mathtt{end}$	l	$\{l\}$

Representing Control Flow II

Definition 2.4 (Flow relation)

Given a statement $c \in Cmd$, the (control) flow relation

$$\mathrm{flow}(c) \subseteq Lab \times Lab$$

is defined by

$$\begin{aligned}
\mathrm{flow}([\mathtt{skip}]^l) &:= \emptyset\\
\mathrm{flow}([x := a]^l) &:= \emptyset\\
\mathrm{flow}(c_1 ; c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \mathrm{final}(c_1) \times \{\mathrm{init}(c_2)\}\\
\mathrm{flow}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \mathtt{end}) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \{(l, \mathrm{init}(c_1)), (l, \mathrm{init}(c_2))\}\\
\mathrm{flow}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c\ \mathtt{end}) &:= \mathrm{flow}(c) \cup \{(l, \mathrm{init}(c))\} \cup \mathrm{final}(c) \times \{l\}
\end{aligned}$$

Representing Control Flow III

Example 2.5

$c =$ [z := 1]^1;

```
while [x > 0]^2 do
  [z := z*y]^3;
  [x := x-1]^4
end
```

Representing Control Flow III

Example 2.5

$c = $ [z := 1]^1;
    while [x > 0]^2 do
      [z := z*y]^3;
      [x := x-1]^4
    end

$$\mathsf{init}(c) = 1$$
$$\mathsf{final}(c) = \{2\}$$
$$\mathsf{flow}(c) = \{(1,2),(2,3),(3,4),(4,2)\}$$

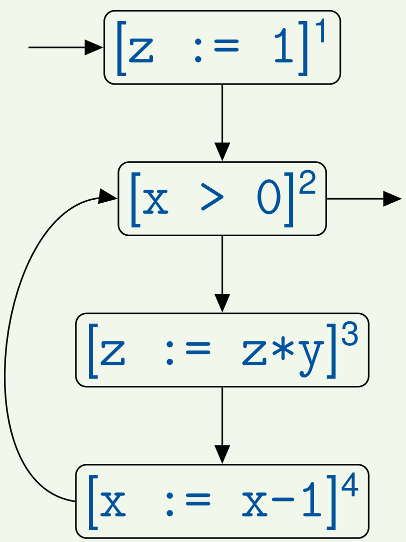
Representing Control Flow III

Example 2.5

```
c = [z := 1]^1;
    while [x > 0]^2 do
      [z := z*y]^3;
      [x := x-1]^4
    end
```

$$\begin{aligned}\mathsf{init}(c) &= 1\\ \mathsf{final}(c) &= \{2\}\\ \mathsf{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\}\end{aligned}$$

Visualisation by (control) flow graph:



Representing Control Flow IV

- To simplify the presentation we will often assume that the program $c \in Cmd$ under consideration has an isolated entry, meaning that
$$\{l \in Lab \mid (l, \text{init}(c)) \in \text{flow}(c)\} = \emptyset$$
(which is the case when c does not start with a `while` loop)

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- Similarly: $c \in Cmd$ has isolated exits if
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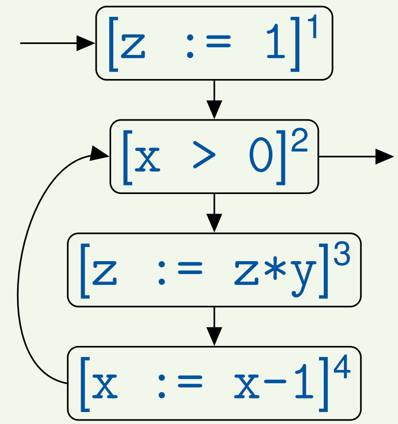
Example 2.6 (cf. Ex. 2.5)



has an isolated entry but not isolated exits

Outline of Lecture 2

An Example: Available Expressions Analysis

Goal of Available Expressions Analysis

Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

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- Can be used for Common Subexpression Elimination:
  replace subexpression by variable that contains up-to-date value
- Only interesting for non-trivial (i.e., complex) arithmetic expressions

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Example 2.7 (Available Expressions Analysis)

$[\texttt{x := a+b}]^1$;
$[\texttt{y := a*b}]^2$;
while $[\texttt{y > a+b}]^3$ do
  $[\texttt{a := a+1}]^4$;
  $[\texttt{x := a+b}]^5$
end

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- a+b available at label 3

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- a+b available at label 3
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[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
end
```

- `a+b` available at label 3
- `a+b` not available at label 5
- possible optimisation: `while [y > x]^3 do`

Formalising Available Expressions Analysis I

- Given $a \in AExp$, $b \in BExp$, $c \in Cmd$,
  - $Var_a/Var_b/Var_c$ denotes the set of all variables occurring in $a/b/c$
  - $CExp_b/CExp_c$ denote the sets of all complex arithmetic expressions occurring in b/c

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- An expression a is killed in a block B if any of the variables in a is modified in B
- Formally: $\mathsf{kill}_{\mathsf{AE}} : Blk_c \to 2^{CExp_c}$ is defined by

$$\begin{aligned}
\mathsf{kill}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset \\
\mathsf{kill}_{\mathsf{AE}}([x := a]^l) &:= \{a' \in CExp_c \mid x \in Var_{a'}\} \\
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- An expression a is generated in a block B if it is evaluated in and none of its variables are modified by B
- Formally: $\mathsf{gen}_{\mathsf{AE}} : Blk_c \to 2^{CExp_c}$ is defined by

$$\begin{aligned} \mathsf{gen}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{gen}_{\mathsf{AE}}([x := a]^l) &:= \{a \mid x \notin Var_a\} \\ \mathsf{gen}_{\mathsf{AE}}([b]^l) &:= CExp_b \end{aligned}$$

Formalising Available Expressions Analysis II

Example 2.8 ($\mathsf{kill}_{AE}/\mathsf{gen}_{AE}$ functions)

```
c = [x := a+b]^1;
    [y := a*b]^2;
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- $CExp_c = \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}$

Formalising Available Expressions Analysis II

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Lab_c	$\mathsf{kill}_{AE}(B^l)$	$\mathsf{gen}_{AE}(B^l)$
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The Equation System I

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An Example: Available Expressions Analysis

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$$\mathrm{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathrm{init}(c) \\ \bigcap\{\varphi_{l'}(\mathrm{AE}_{l'}) \mid (l', l) \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

where $\varphi_{l'} : 2^{CExp_c} \to 2^{CExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathrm{kill}_{\mathrm{AE}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{AE}}(B^{l'})$$

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- Characterisation of analysis:
  flow-sensitive: results depending on order of assignments
  forward: starts in $\mathrm{init}(c)$ and proceeds downwards
  must: $\bigcap$ in equations for AE_l
- Later: solution not necessarily unique
  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

$$\mathrm{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathrm{init}(c) \\ \bigcap \{\varphi_{l'}(\mathrm{AE}_{l'}) \mid (l', l) \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

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Example 2.9 (AE equation system)

$c =$ [x := a+b]^1;
[y := a*b]^2;
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Equations:

$$
\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}
\end{aligned}
$$

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\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{a+b\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{a+b, a*b, a+1\}
\end{aligned}$$

(Unique) solution:
$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \{a+b\} \\
\mathsf{AE}_3 &= \{a+b\} \\
\mathsf{AE}_4 &= \{a+b\} \\
\mathsf{AE}_5 &= \emptyset
\end{aligned}$$

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Another Example: Live Variables Analysis

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- All variables considered to be live at the end of the program (alternative: restriction to output variables)

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- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re-define the variable
- All variables considered to be live at the end of the program (alternative: restriction to output variables)
- Can be used for Dead Code Elimination: remove assignments to non-live variables

An Example

Example 2.10 (Live Variables Analysis)

```
[x := 2]^1;
[y := 4]^2;
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An Example

Example 2.10 (Live Variables Analysis)

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- x not live at exit from label 1
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An Example

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end;
[x := z]7
```

- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3

An Example

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```
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end;
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- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3
- z live at exits from 5 and 6

An Example

Example 2.10 (Live Variables Analysis)

```
[x := 2]¹;
[y := 4]²;
[x := 1]³;
if [y > 0]⁴ then
  [z := x]⁵
else
  [z := y*y]⁶
end;
[x := z]⁷
```

- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3
- z live at exits from 5 and 6
- possible optimisation: remove $[\texttt{x := 2}]^1$

Formalising Live Variables Analysis I

- A variable on the left-hand side of an assignment is killed by the assignment; tests and `skip` do not kill
- Formally: $\mathsf{kill}_{\mathsf{LV}} : Blk_c \rightarrow 2^{Var_c}$ is defined by

$$\begin{aligned} \mathsf{kill}_{\mathsf{LV}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{kill}_{\mathsf{LV}}([x := a]^l) &:= \{x\} \\ \mathsf{kill}_{\mathsf{LV}}([b]^l) &:= \emptyset \end{aligned}$$

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- Every reading access generates a live variable
- Formally: $\mathsf{gen}_{\mathsf{LV}} : Blk_c \to 2^{Var_c}$ is defined by

$$\begin{aligned} \mathsf{gen}_{\mathsf{LV}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{gen}_{\mathsf{LV}}([x := a]^l) &:= Var_a \\ \mathsf{gen}_{\mathsf{LV}}([b]^l) &:= Var_b \end{aligned}$$

Formalising Live Variables Analysis II

Example 2.11 ($kill_{LV}$/gen_{LV} functions)

$c =$
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- $Var_c = \{x, y, z\}$

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The Equation System I

- For each $l \in Lab_c$, $\mathrm{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l

Another Example: Live Variables Analysis

The Equation System I

- For each $l \in Lab_c$, $\mathrm{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l
- Formally, for a program $c \in Cmd$ with isolated exits:

$$\mathrm{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathrm{final}(c) \\ \bigcup\{\varphi_{l'}(\mathrm{LV}_{l'}) \mid (l, l') \in \mathrm{flow}(c)\} & \text{otherwise} \end{cases}$$

where $\varphi_{l'} : 2^{Var_c} \to 2^{Var_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(V) := (V \setminus \mathrm{kill}_{\mathrm{LV}}(B^{l'})) \cup \mathrm{gen}_{\mathrm{LV}}(B^{l'})$$

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- Characterisation of analysis:

flow-sensitive: results depending on order of assignments

backward: starts in $\mathsf{final}(c)$ and proceeds upwards

may: $\bigcup$ in equations for LV_l

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- Characterisation of analysis:

  flow-sensitive: results depending on order of assignments

  backward: starts in $\mathrm{final}(c)$ and proceeds upwards

  may: $\bigcup$ in equations for LV_l
- Later: solution not necessarily unique

  $\Longrightarrow$ choose least one

Another Example: Live Variables Analysis

The Equation System II

Reminder:

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Example 2.12 (LV equation system)

$c =$
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c = [x := 2]^1;
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Equations:

$$\begin{aligned}
\mathsf{LV}_1 &= \varphi_2(\mathsf{LV}_2) = \mathsf{LV}_2 \setminus \{y\} \\
\mathsf{LV}_2 &= \varphi_3(\mathsf{LV}_3) = \mathsf{LV}_3 \setminus \{x\} \\
\mathsf{LV}_3 &= \varphi_4(\mathsf{LV}_4) = \mathsf{LV}_4 \cup \{y\} \\
\mathsf{LV}_4 &= \varphi_5(\mathsf{LV}_5) \cup \varphi_6(\mathsf{LV}_6) \\
&= ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\}) \\
\mathsf{LV}_5 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\} \\
\mathsf{LV}_6 &= \varphi_7(\mathsf{LV}_7) = (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\} \\
\mathsf{LV}_7 &= \{x, y, z\}
\end{aligned}$$

The Equation System II

Reminder: $\mathsf{LV}_l = \begin{cases} Var_c & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l,l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$

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\end{aligned}$$

(Unique) solution:

$$\begin{aligned}
\mathsf{LV}_1 &= \emptyset \\
\mathsf{LV}_2 &= \{y\} \\
\mathsf{LV}_3 &= \{x, y\} \\
\mathsf{LV}_4 &= \{x, y\} \\
\mathsf{LV}_5 &= \{y, z\} \\
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Outline of Lecture 2

Heading for a Dataflow Analysis Framework

Similarities Between Analysis Problems

- **Observation:** the analyses presented so far have some similarities

$\Rightarrow$ Look for underlying framework

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- **Advantages:**
  - possibility for designing (efficient) generic algorithms for solving dataflow equations
  - enables generic correctness proofs of analyses and algorithms

Similarities Between Analysis Problems

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$\Rightarrow$ Look for underlying framework

- **Advantages:**
  - possibility for designing (efficient) generic algorithms for solving dataflow equations
  - enables generic correctness proofs of analyses and algorithms
- **Overall pattern:** for $c \in Cmd$ and $l \in Lab_c$, the analysis information (AI) is described by equations of the form

$$\mathrm{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathrm{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

  where
  - the set of extremal labels, E, is $\{\mathrm{init}(c)\}$ or $\mathrm{final}(c)$
  - ι specifies the extremal analysis information
  - the combination operator, $\bigsqcup$, is $\bigcap$ or $\bigcup$
  - $\varphi_{l'}$ denotes the transfer function of block $B^{l'}$
  - the flow relation F is $\mathrm{flow}(c)$ or $\mathrm{flow}^R(c)$ $(:= \{(l', l) \mid (l, l') \in \mathrm{flow}(c)\})$

Characterisation of Analyses

Direction of information flow

- **Forward**:
  - $F = \text{flow}(c)$
  - AI_l refers to entry of B^l
  - c has isolated entry
- **Backward**:
  - $F = \text{flow}^R(c)$
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  - c has isolated exits

Quantification over paths

- **May**:
  - $\bigsqcup = \bigcup$
  - property satisfied by some path
  - interested in least solution (later)
- **Must**:
  - $\bigsqcup = \bigcap$
  - property satisfied by all paths
  - interested in greatest solution (later)

Heading for a Dataflow Analysis Framework

Solutions as Fixpoints

- **Wanted:** solution of (dataflow) equation system

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- **Idea:** characterise solution as fixpoint of transformation:

$$(\mathsf{AI}_l = \tau_l)_{l \in Lab_c} \iff \Phi((\mathsf{AI}_l)_{l \in Lab_c}) = (\mathsf{AI}_l)_{l \in Lab_c}$$

where $\Phi\left((\mathsf{AI}_l)_{l \in Lab_c}\right) := (\tau_l)_{l \in Lab_c}$

Solutions as Fixpoints

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  where $\Phi\left((\mathsf{AI}_l)_{l \in Lab_c}\right) := (\tau_l)_{l \in Lab_c}$
- **Approach:** approximate fixpoint by iteration

Heading for a Dataflow Analysis Framework

Roadmap

Goal: solve dataflow equation system by fixpoint iteration

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Heading for a Dataflow Analysis Framework

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Heading for a Dataflow Analysis Framework

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5. Guarantee termination of fixpoint iteration by ascending chain condition
6. Optimise fixpoint iteration by worklist algorithm