Static Program Analysis

Lecture 14: Abstract Interpretation V (Predicate Abstraction)

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Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/teaching/ss-18/spa/

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

Outline of Lecture 14

Recap: Abstraction of 16-Bit Multiplication

Abstract Interpretation of Multiplier

Example (Abstraction of 16-bit multiplier; cf. Example 13.3)

Abstract execution for

- $\texttt{f1} = 101_2\ (= 5)$, $\texttt{f2} = 1001010_2\ (= 74)$
- $\texttt{out}$, $\texttt{ovf}$ with arbitrary initial values
- $m = 5$

$$\vdots$$
$$\Downarrow$$
$$\langle [(\texttt{ovf},\texttt{out}) := (\texttt{out} : 17) + \texttt{f2}]^5; \ldots, \{[\texttt{f1} \mapsto 101_2, \texttt{f2} \mapsto 4, \texttt{out} \mapsto 0, \texttt{ovf} \mapsto 0]\}\rangle$$
$$\Downarrow$$
$$\langle \ldots, \{[\texttt{f1} \mapsto 101_2, \texttt{f2} \mapsto 4, \texttt{out} \mapsto 4, \texttt{ovf} \mapsto 0], \rangle$$
$$[\texttt{f1} \mapsto 101_2, \texttt{f2} \mapsto 4, \texttt{out} \mapsto 3, \texttt{ovf} \mapsto 1]\}$$
$$\Downarrow$$
$$\vdots$$

since $\texttt{out}' = (\texttt{out} + \texttt{f2} - 2^{16}) \bmod 5 = (0 + 4 - 1) \bmod 5 = 3$ in case of overflow

Recap: Abstraction of 16-Bit Multiplication

Extending the Specification

- **Observation:** the soundness property

$$\langle c, \sigma \rangle \to^+ \langle \downarrow, \sigma' \rangle \implies \sigma'(\texttt{out}) = \sigma(\texttt{f1}) \cdot \sigma(\texttt{f2}) \vee \sigma'(\texttt{ovf}) = 1$$

  is easily satisfied by always setting $\sigma'(\texttt{ovf}) = 1$
- Additionally required to ensure correctness in absence of overflows: reasoning about sizes (of bit string representations) of numbers

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Logarithmic abstraction

- For $n \in \mathbb{N}$, let $s(n) := \lceil \log_2(n+1) \rceil$ be the minimal number of bits required to represent n (where $s(n) = 0$)
- It holds: $s(m \cdot n) \begin{cases} \leq 16 & \text{if } s(m) + s(n) \leq 16 \\ \geq 17 & \text{if } s(m) + s(n) \geq 18 \end{cases}$
- Strengthened (and satisfied) specification: if $\langle c, \sigma \rangle \to^+ \langle \downarrow, \sigma' \rangle$, then

$$\begin{cases} \sigma'(\texttt{out}) = \sigma(\texttt{f1}) \cdot \sigma(\texttt{f2}) \wedge \sigma'(\texttt{ovf}) = 0 & \text{if } s(\sigma(\texttt{f1})) + s(\sigma(\texttt{f2})) \leq 16 \\ \sigma'(\texttt{ovf}) = 1 & \text{if } s(\sigma(\texttt{f1})) + s(\sigma(\texttt{f2})) \geq 18 \\ \sigma'(\texttt{out}) = \sigma(\texttt{f1}) \cdot \sigma(\texttt{f2}) \vee \sigma'(\texttt{ovf}) = 1 & \text{otherwise} \end{cases}$$

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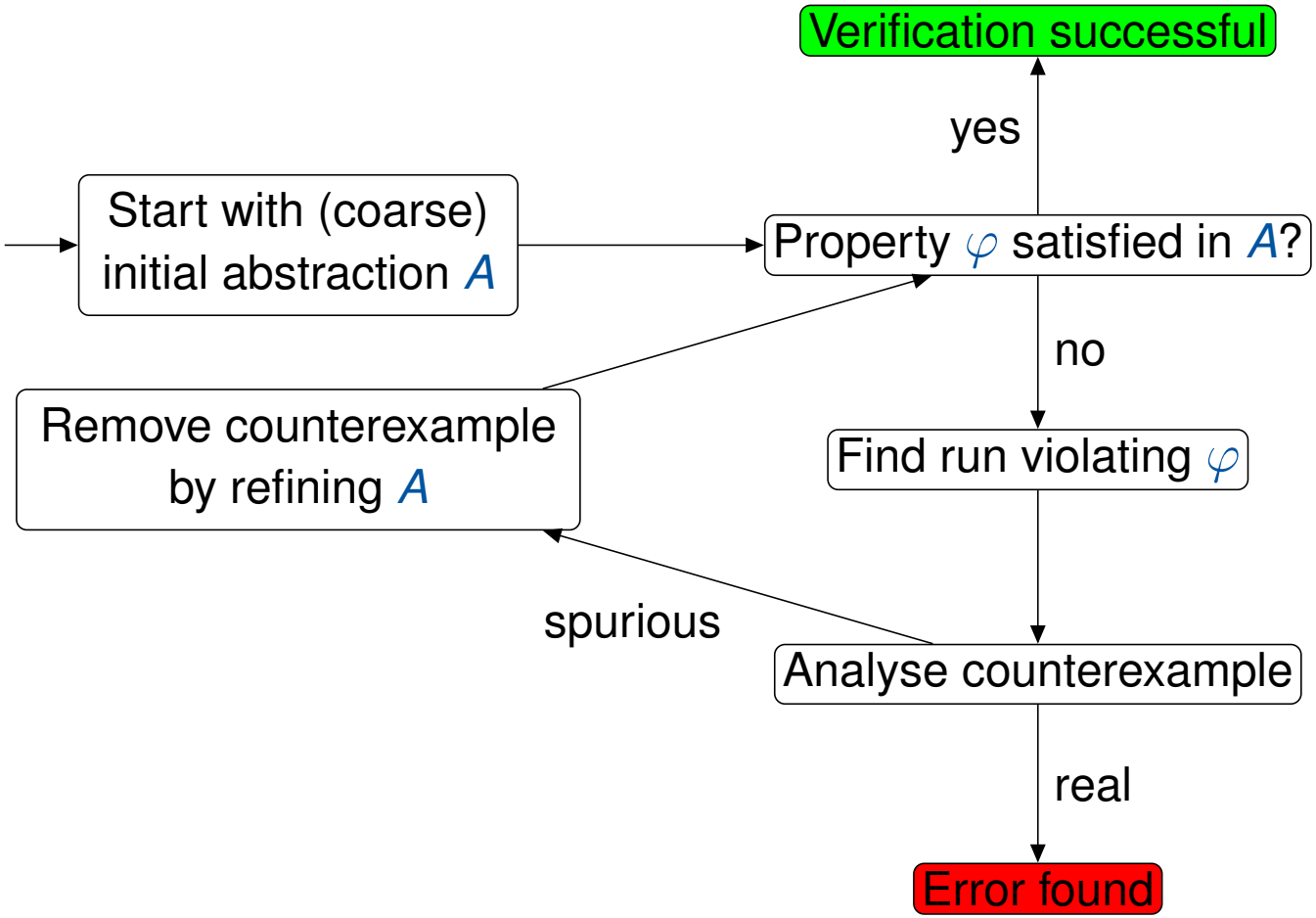
- **Abstraction refinement:** most successful (automatic) method based on
  - predicate abstraction and
  - analysing counterexamples
- **Difference** to standard abstract interpretation with fixed domain:
  abstraction parametrised by and specific to program

Counterexample-Guided Abstraction Refinement (CEGAR)



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2. Use Galois connection that classifies program states according to validity of predicates (predicate abstraction)
3. Compute new abstract semantics and search for new counterexamples
4. Iterate until property satisfied or real counterexample found
   (with increasing set of predicates; can entail non-termination)

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- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(P) := \left(\left\{ \bigwedge Q \mid Q \subseteq P \cup \neg P \right\} \cup \{\text{false}\}, \models \right).$$

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Abbreviation: $\text{true} := \bigwedge \emptyset$ (and $\text{false} \equiv \bigwedge \{p_i, \neg p_i, \ldots\}$ if $P \neq \emptyset$)

Predicate Abstraction II

Lemma 14.2

$Abs(P)$ is a complete lattice with

- $\bot = \text{false}$, $\top = \text{true}$
- $Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2}$ *where* $\overline{b} := \bigwedge \{q \in P \cup \neg P \mid b \models q\}$
  (i.e., $\overline{b}$ is strongest formula in $Abs(P)$ that is implied by b)
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Example 14.3

Let $P := \{p_1, p_2, p_3\}$ with $p_1 := (\mathrm{x} = 1)$, $p_2 := (\mathrm{y} = 2)$, $p_1 := (\mathrm{z} = 3)$ (indep.).

1. For $Q_1 := p_1 \wedge \neg p_2$ and $Q_2 := \neg p_2 \wedge p_3$, we obtain

$$Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge \neg p_2 \wedge p_3 \qquad (= \bigwedge(Q_1 \cup Q_2))$$
$$Q_1 \sqcup Q_2 = \overline{Q_1 \vee Q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} = \neg p_2 \qquad (= \bigwedge(Q_1 \cap Q_2))$$

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Predicate Abstraction III

Lemma 14.4

1. *If predicates in P are pairwise independent, then*
   - $Q_1 \sqcap Q_2 \left(= \overline{Q_1 \wedge Q_2}\right) = \bigwedge(Q_1 \cup Q_2)$
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   $\Rightarrow Q_1 \sqcap Q_2 = \overline{Q_1 \wedge Q_2} = p_1 \wedge p_2 \wedge p_3 \not\equiv \bigwedge(Q_1 \cup Q_2) = p_1 \wedge p_2$

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Note: If interdependencies are ignored, then (computationally simpler) Cartesian Abstraction is performed.

Predicate Abstraction IV

Definition 14.6 (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\alpha : 2^{\Sigma} \to \mathit{Abs}(P) \qquad \gamma : \mathit{Abs}(P) \to 2^{\Sigma}$$

with $\alpha(S) := \bigsqcup\{Q_\sigma \mid \sigma \in S\}$ $\qquad \gamma(Q) := \{\sigma \in \Sigma \mid \sigma \models Q\}$

where $Q_\sigma := \bigwedge(\{p \in P \mid \sigma \models p\} \cup \{\neg p \in \neg P \mid \sigma \not\models p\})$

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Example 14.7

- Let $\mathit{Var} := \{\mathtt{x}, \mathtt{y}\}$ and $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x} <= \mathtt{y})$, $p_2 := (\mathtt{x} = \mathtt{y})$, $p_3 := (\mathtt{x} > \mathtt{y})$

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- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$, then

$$\begin{aligned}\alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3) \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3\end{aligned}$$

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$$\begin{aligned}\alpha(S) &= Q_{\sigma_1} \sqcup Q_{\sigma_2} \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3) \\ &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3\end{aligned}$$
- If $Q = p_1 \wedge \neg p_2 \in \mathit{Abs}(P)$, then $\gamma(Q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

Outline of Lecture 14

Abstract Semantics for Predicate Abstraction

Definition 14.8 (Execution relation for predicate abstraction)

If $c \in \mathit{Cmd}$ and $Q \in \mathit{Abs}(P)$, then $\langle c, Q \rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, Q \rangle \Rightarrow \langle \downarrow, Q \rangle} \qquad \text{(asgn)}\ \frac{}{\langle x := a, Q \rangle \Rightarrow \langle \downarrow, \bigsqcup \{ Q_{\sigma[x \mapsto \mathit{val}_\sigma(a)]} \mid \sigma \models Q \} \rangle}$$

$$\text{(seq1)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle c_1', Q' \rangle \quad c_1' \neq \downarrow}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_1' ; c_2, Q' \rangle} \qquad \text{(seq2)}\ \frac{\langle c_1, Q \rangle \Rightarrow \langle \downarrow, Q' \rangle}{\langle c_1 ; c_2, Q \rangle \Rightarrow \langle c_2, Q' \rangle}$$

$$\text{(if1)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, Q \rangle \Rightarrow \langle c_1, \overline{Q \wedge b} \rangle}$$

$$\text{(if2)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}, Q \rangle \Rightarrow \langle c_2, \overline{Q \wedge \neg b} \rangle}$$

$$\text{(wh1)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, Q \rangle \Rightarrow \langle c ; \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, \overline{Q \wedge b} \rangle}$$

$$\text{(wh2)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}, Q \rangle \Rightarrow \langle \downarrow, \overline{Q \wedge \neg b} \rangle}$$

Abstract Semantics for Predicate Abstraction

Remarks

- In Rule (asgn), $\bigsqcup\{Q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models Q\}$ denotes the strongest postcondition of Q w.r.t. statement $x := a$. It covers all states that are obtained from a state satisfying Q by applying the assignment $x := a$:

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- If $P = \emptyset$ (and thus $Abs(P) = \{\text{true}, \text{false}\}$) and if no $b \in BExp_c$ is a tautology or contradiction (i.e., resp. equivalent to true or false), then the abstract transition system with initial configuration $\langle c, \text{true}\rangle$ corresponds to the control flow graph of c

Abstract Semantics for Predicate Abstraction

An Example

Example 14.9

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if [x > y]^1 then
  while [¬(y = 0)]^2 do
    [x := x - 1;]^3;
    [y := y - 1;]^4
  end;
  if [x > y]^5 then
    [skip]^6
  else
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- **Abstract transition system:** on the board
- **Remark:** $p_1 := (\texttt{x > y})$ alone not sufficient to prove loop invariant
  (as not necessarily valid after label 3)

Outline of Lecture 14

Computation of Postconditions

Problem: $\overline{b} = \bigwedge \{ q \in P \cup \neg P \mid b \models q \}$ (i.e., the strongest formula in $Abs(P)$ that is implied by b) is generally not computable (due to undecidability of implication in certain logics)

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- Restriction to finite domains:
  - for example, binary numbers of fixed size
  - thus everything (domain, Galois connection, ...) exactly computable
  - problem: exponential blowup in n $\Longrightarrow$ solution: Binary Decision Diagrams (BDDs)