Overview

Introduction

Modelling parallel systems

Linear Time Properties

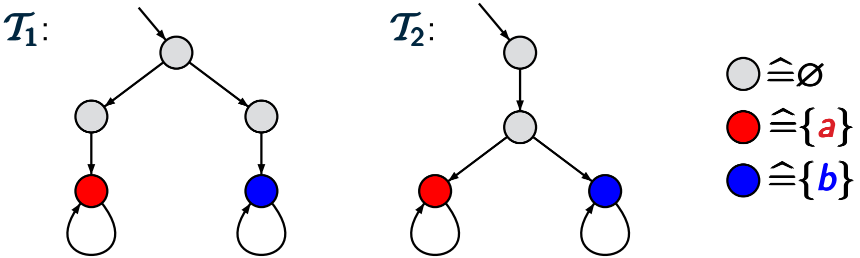
Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

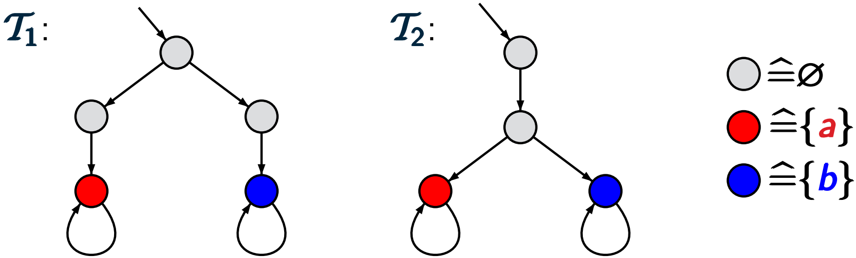
Equivalences and Abstraction

Trace equivalence



$$Traces(\mathcal{T}_1) = \{ \varnothing \varnothing a^\omega, \ \varnothing \varnothing b^\omega \} = Traces(\mathcal{T}_1)$$

Trace equivalence

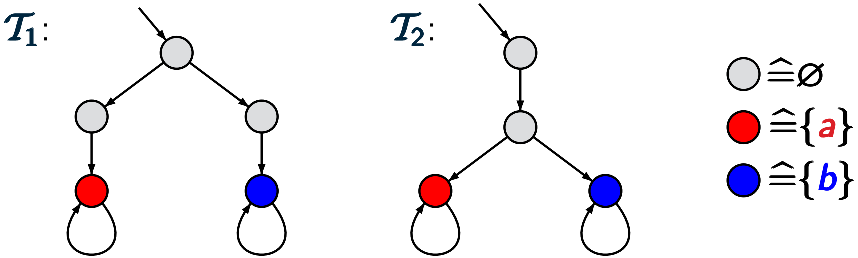


$$Traces(\mathcal{T}_1) = \{\ \varnothing\,\varnothing\, a^\omega,\ \varnothing\,\varnothing\, b^\omega\ \} = Traces(\mathcal{T}_1)$$

CTL-formula $\Phi = \exists\bigcirc(\exists\bigcirc a \wedge \exists\bigcirc b)$

$\mathcal{T}_1 \not\models \Phi$ and $\mathcal{T}_2 \models \Phi$

Trace equivalence is not compatible with CTL



$$\mathit{Traces}(\mathcal{T}_1) = \{ \varnothing \varnothing a^\omega, \ \varnothing \varnothing b^\omega \} = \mathit{Traces}(\mathcal{T}_1)$$

CTL-formula $\Phi = \exists \bigcirc \big(\exists \bigcirc a \wedge \exists \bigcirc b \big)$

$\mathcal{T}_1 \not\models \Phi$ and $\mathcal{T}_2 \models \Phi$

Implementation relations

- for the design of complex systems

  $\rightsquigarrow$ comparison of **2** transition systems

- for the analysis of complex systems

  $\rightsquigarrow$ homogeneous model checking approach

  $\rightsquigarrow$ graph minimization

  use equivalence relation $\sim$ for the states of a single transition system $\mathcal{T}$ and analyze the quotient $\mathcal{T}/\!\sim$

goal: define the equivalence $\sim$ in such a way that

$$\mathcal{T} \models \Phi \quad \text{iff} \quad \mathcal{T}/\!\sim\, \models \Phi$$

for all "relevant" properties Φ

Linear-time implementation relations

Linear-time implementation relations

finite trace inclusion and equivalence:

e.g., $\mathit{Tracesfin}(\mathcal{T}_1) \subseteq \mathit{Tracesfin}(\mathcal{T}_2)$

trace inclusion and trace equivalence:

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Linear-time implementation relations

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* none of the LT relations is compatible with **CTL**

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preserves all **LTL** properties

* none of the LT relations is compatible with **CTL**
* checking LT relations is computationally hard

Classification of implementation relations

- linear vs. branching time
  - linear time: trace relations
  - branching time: (bi)simulation relations
- (nonsymmetric) preorders vs. equivalences:
  - preorders: trace inclusion, simulation
  - equivalences: trace equivalence, bisimulation
- strong vs. weak relations
  - strong: reasoning about all transitions
  - weak: abstraction from stutter steps

Bisimulation for two transition systems

let $\mathcal{T}_1 = (S_1, \cancel{Act_1}, \to_1, S_{0,1}, AP, L_1)$,

$\mathcal{T}_2 = (S_2, \cancel{Act_2}, \to_2, S_{0,2}, AP, L_2)$

be two transition systems

- with the same set AP $\longleftarrow$ $\boxed{\text{observables}}$
- possibly containing terminal states

> Bisimulation equivalence of $\mathcal{T}_1$ and $\mathcal{T}_2$ requires that $\mathcal{T}_1$ and $\mathcal{T}_2$ can simulate each other in a stepwise manner.

Bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$

binary relation $\mathcal{R} \subseteq S_1 \times S_2$ s.t. for all $(s_1, s_2) \in \mathcal{R}$:

(1) $L_1(s_1) = L_2(s_2)$

(2) $\forall s_1' \in Post(s_1)\, \exists s_2' \in Post(s_2)$ s.t. $(s_1', s_2') \in \mathcal{R}$

$$\boxed{\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[d] & s_2 \\ s_1' & \end{tikzcd}} \quad \text{can be completed to} \quad \boxed{\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[d] & s_2 \arrow[d] \\ s_1' \arrow[r, dash, "\mathcal{R}"] & s_2' \end{tikzcd}}$$

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$$\begin{tikzcd} s_1 \arrow[r, phantom, "-\mathcal{R}-"] \arrow[d] & s_2 \\ s_1' & \end{tikzcd} \quad \text{can be completed to} \quad \begin{tikzcd} s_1 \arrow[r, phantom, "-\mathcal{R}-"] \arrow[d] & s_2 \arrow[d] \\ s_1' \arrow[r, phantom, "-\mathcal{R}-"] & s_2' \end{tikzcd}$$

(3) $\forall s_2' \in Post(s_2) \, \exists s_1' \in Post(s_1)$ s.t. $(s_1', s_2') \in \mathcal{R}$

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$$\boxed{\begin{array}{ccc} s_1 & -\mathcal{R}- & s_2 \\ \downarrow & & \\ s_1' & & \end{array}} \quad \text{can be completed to} \quad \boxed{\begin{array}{ccc} s_1 & -\mathcal{R}- & s_2 \\ \downarrow & & \downarrow \\ s_1' & -\mathcal{R}- & s_2' \end{array}}$$

(3) $\forall s_2' \in Post(s_2)\, \exists s_1' \in Post(s_1)$ s.t. $(s_1', s_2') \in \mathcal{R}$

and such that the following initial condition holds:

(I) $\forall s_{0,1} \in S_{0,1}\, \exists s_{0,2} \in S_{0,2}$ s.t. $(s_{0,1}, s_{0,2}) \in \mathcal{R}$

$\forall s_{0,2} \in S_{0,2}\, \exists s_{0,1} \in S_{0,1}$ s.t. $(s_{0,1}, s_{0,2}) \in \mathcal{R}$

Bisimulation equivalence $\sim$

bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$: relation $\mathcal{R} \subseteq S_1 \times S_2$ s.t.

for all $(s_1, s_2) \in \mathcal{R}$: (1) labeling condition

$\left.\begin{array}{l}(2)\\(3)\end{array}\right\}$ mutual stepwise simulation

and initial condition (I)

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> bisimulation equivalence $\sim$ for TS:
>
> $\mathcal{T}_1 \sim \mathcal{T}_2$ iff there is a bisimulation $\mathcal{R}$ for $(\mathcal{T}_1, \mathcal{T}_2)$

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for all $(s_1, s_2) \in \mathcal{R}$: (1) labeling condition

(2), (3) mutual stepwise simulation

and initial condition (I)

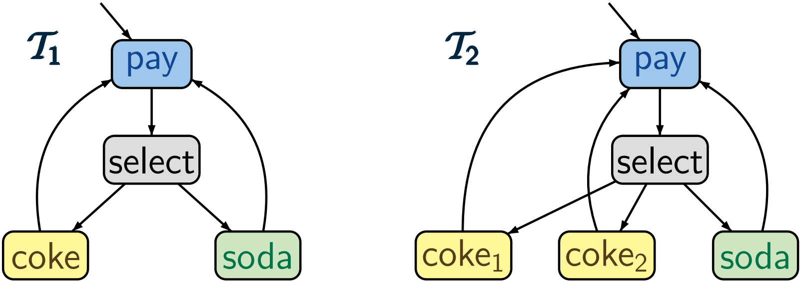
bisimulation equivalence $\sim$ for TS:

$\mathcal{T}_1 \sim \mathcal{T}_2$ iff there is a bisimulation $\mathcal{R}$ for $(\mathcal{T}_1, \mathcal{T}_2)$

for state s_1 of $\mathcal{T}_1$ and state s_2 of $\mathcal{T}_2$:

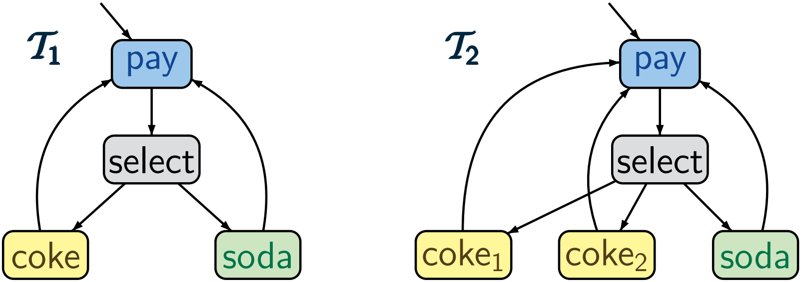
$s_1 \sim s_2$ iff there exists a bisimulation $\mathcal{R}$ for $(\mathcal{T}_1, \mathcal{T}_2)$ such that $(s_1, s_2) \in \mathcal{R}$

Two beverage machines



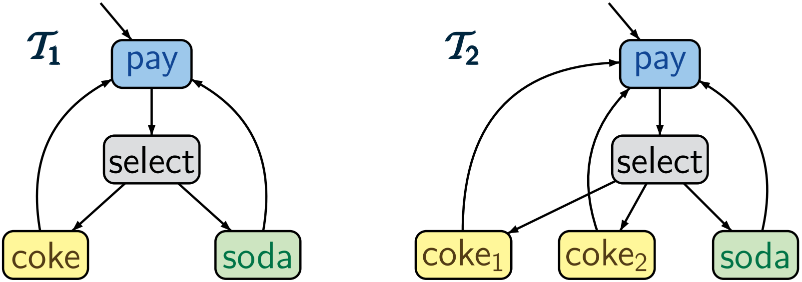
$$AP = \{pay, coke, soda\}$$

Two beverage machines



$AP = \{pay, coke, soda\}$

$\mathcal{T}_1 \sim \mathcal{T}_2$ as there is a bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$:

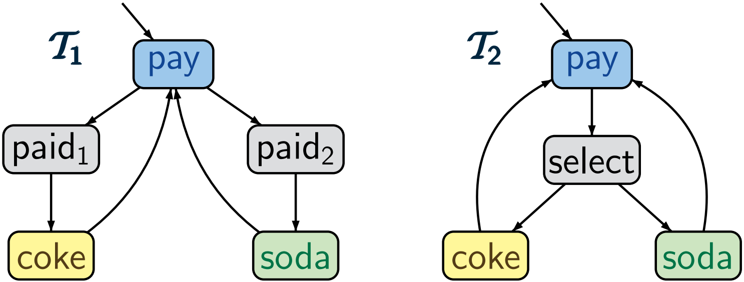


$AP = \{pay, coke, soda\}$

$\mathcal{T}_1 \sim \mathcal{T}_2$ as there is a bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$:

$\{$ (pay,pay), (select,select), (soda,soda)
(coke,coke$_1$), (coke,coke$_2$) $\}$

Two beverage machines



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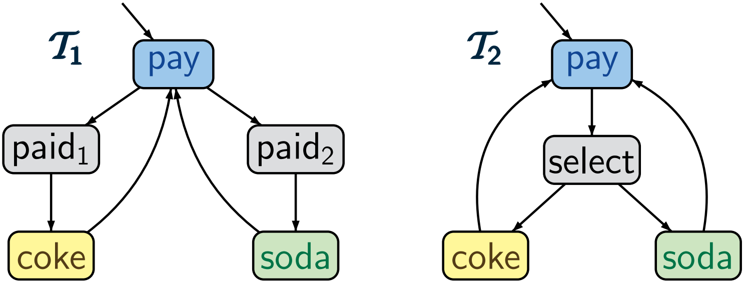

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$\mathcal{T}_1 \not\sim \mathcal{T}_2$

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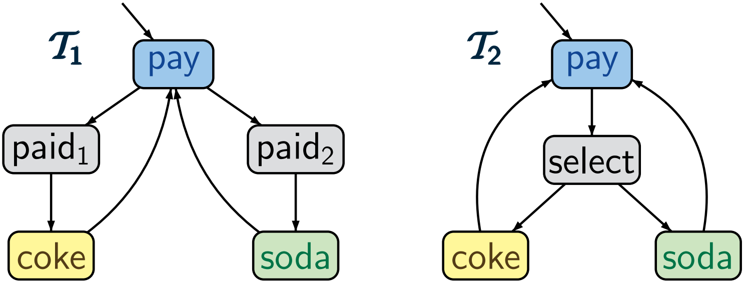

$AP = \{pay, coke, soda\}$

$\mathcal{T}_1 \not\sim \mathcal{T}_2$

because there is no state in $\mathcal{T}_1$ that has both

- a successor labeled with coke and
- a successor labeled with soda

Simulation condition of bisimulations

$$\boxed{\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[d] & s_2 \\ s_1' & \end{tikzcd}}$$

can be completed to

$$\boxed{\begin{tikzcd} s_1 \arrow[r, dash, "\mathcal{R}"] \arrow[d] & s_2 \arrow[d] \\ s_1' \arrow[r, dash, "\mathcal{R}"] & s_2' \end{tikzcd}}$$

Path lifting for bisimulation $\mathcal{R}$

$$
\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \\
s_{1,1} & & \\
\downarrow & & \\
s_{1,2} & & \\
\downarrow & & \\
s_{1,3} & & \\
\downarrow & & \\
s_{1,4} & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

Path lifting for bisimulation $\mathcal{R}$

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\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
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s_{1,1} & & \\
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s_{1,3} & & \\
\downarrow & & \\
s_{1,4} & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

can be completed to

Path lifting for bisimulation $\mathcal{R}$

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$$
\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \\
s_{1,1} & & \\
\downarrow & & \\
s_{1,2} & & \\
\downarrow & & \\
s_{1,3} & & \\
\downarrow & & \\
s_{1,4} & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

can be completed to

$$
\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \downarrow \\
s_{1,1} & -\mathcal{R}- & s_{2,1} \\
\downarrow & & \\
s_{1,2} & & \\
\downarrow & & \\
s_{1,3} & & \\
\downarrow & & \\
s_{1,4} & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

Path lifting for bisimulation $\mathcal{R}$

$$
\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \\
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\downarrow & & \\
\vdots & &
\end{array}
$$

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s_1 & -\mathcal{R}- & s_2 \\
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s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \\
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s_{1,1} & -\mathcal{R}- & s_{2,1} \\
\downarrow & & \downarrow \\
s_{1,2} & -\mathcal{R}- & s_{2,2} \\
\downarrow & & \downarrow \\
s_{1,3} & -\mathcal{R}- & s_{2,3} \\
\downarrow & & \\
s_{1,4} & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

Path lifting for bisimulation $\mathcal{R}$

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\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \\
s_{1,1} & & \\
\downarrow & & \\
s_{1,2} & & \\
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s_{1,3} & & \\
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\downarrow & & \\
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\qquad \text{can be completed to} \qquad
\begin{array}{ccc}
s_1 & -\mathcal{R}- & s_2 \\
\downarrow & & \downarrow \\
s_{1,1} & -\mathcal{R}- & s_{2,1} \\
\downarrow & & \downarrow \\
s_{1,2} & -\mathcal{R}- & s_{2,2} \\
\downarrow & & \downarrow \\
s_{1,3} & -\mathcal{R}- & s_{2,3} \\
\downarrow & & \downarrow \\
s_{1,4} & -\mathcal{R}- & s_{2,4} \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

Properties of bisimulation equivalence

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- reflexivity: $\mathcal{T} \sim \mathcal{T}$ for all transition systems $\mathcal{T}$

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$\uparrow$

If S is the state space of $\mathcal{T}$ then

$$\mathcal{R} = \{(s,s) : s \in S\}$$

is a bisimulation for $(\mathcal{T}, \mathcal{T})$

Properties of bisimulation equivalence

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- reflexivity: $\mathcal{T} \sim \mathcal{T}$ for all transition systems $\mathcal{T}$
- symmetry: $\mathcal{T}_1 \sim \mathcal{T}_2$ implies $\mathcal{T}_2 \sim \mathcal{T}_1$

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If $\mathcal{R}$ is a bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$ then

$$\mathcal{R}^{-1} = \{(s_2, s_1) : (s_1, s_2) \in \mathcal{R}\}$$

is a bisimulation for $(\mathcal{T}_2, \mathcal{T}_1)$

Properties of bisimulation equivalence

$\sim$ is an equivalence, i.e.,

- reflexivity: $\mathcal{T} \sim \mathcal{T}$ for all transition systems $\mathcal{T}$
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- transitivity: if $\mathcal{T}_1 \sim \mathcal{T}_2$ and $\mathcal{T}_2 \sim \mathcal{T}_3$ then $\mathcal{T}_1 \sim \mathcal{T}_3$

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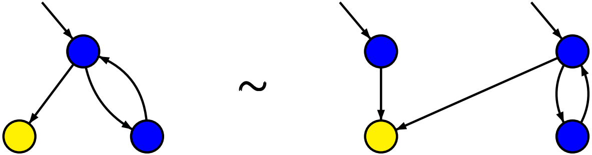
- transitivity: if $\mathcal{T}_1 \sim \mathcal{T}_2$ and $\mathcal{T}_2 \sim \mathcal{T}_3$ then $\mathcal{T}_1 \sim \mathcal{T}_3$

Let $\mathcal{R}_{1,2}$ be a bisimulation for $(\mathcal{T}_1, \mathcal{T}_2)$,
$\mathcal{R}_{2,3}$ be a bisimulation for $(\mathcal{T}_2, \mathcal{T}_3)$.

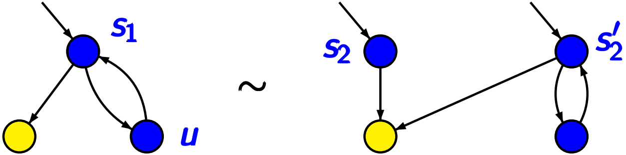
$$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_3) : \exists s_2 \text{ s.t. } (s_1, s_2) \in \mathcal{R}_{1,2} \text{ and } (s_2, s_3) \in \mathcal{R}_{2,3} \}$$

is a bisimulation for $(\mathcal{T}_1, \mathcal{T}_3)$

Correct or wrong?



Correct or wrong?



wrong

$s_1 \longrightarrow u$, but $s_2 \not\longrightarrow blue$ (thus $s_1 \not\sim s_2$)

Correct or wrong?

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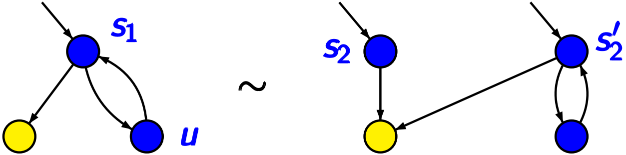

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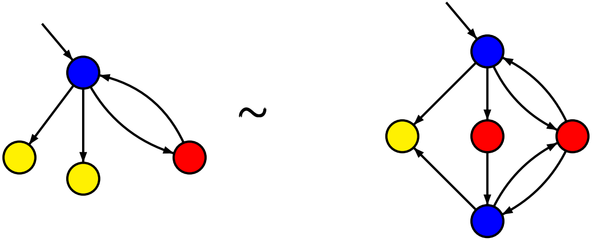
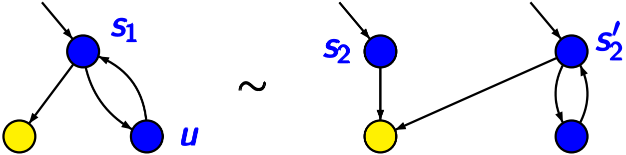


Correct or wrong?



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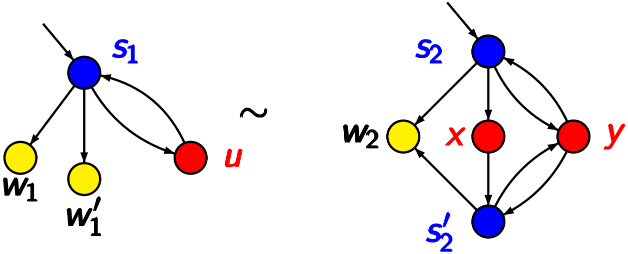
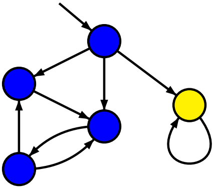


correct

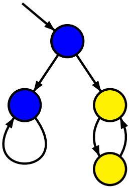
bisimulation:

$\{(w_1, w_2), (w_1', w_2), (s_1, s_2), (s_1, s_2'), (u, x), (u, y)\}$

Correct or wrong?

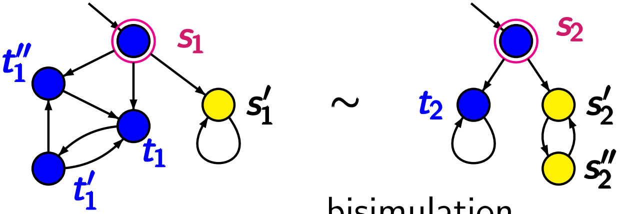


$\sim$



correct

Correct or wrong?

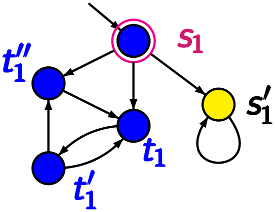


correct

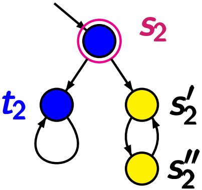
bisimulation

$$\{(s_1, s_2), (s'_1, s'_2), (s'_1, s''_2), (t_1, t_2), (t'_1, t_2), (t''_1, t_2)\}$$

Correct or wrong?



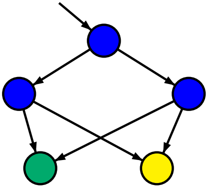
$\sim$



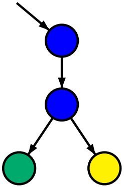
correct

bisimulation

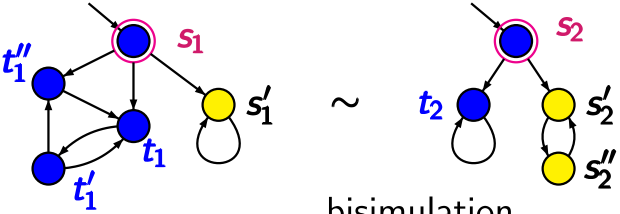
$$\{(s_1, s_2), (s_1', s_2'), (s_1', s_2''), (t_1, t_2), (t_1', t_2), (t_1'', t_2)\}$$



$\sim$



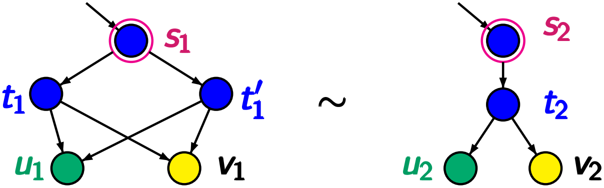
Correct or wrong?



correct

bisimulation

$$\{(s_1, s_2), (s_1', s_2'), (s_1', s_2''), (t_1, t_2), (t_1', t_2), (t_1'', t_2)\}$$



correct

bisimulation: $\{(s_1, s_2), (t_1, t_2), (t_1', t_2), (u_1, u_2), (v_1, v_2)\}$

Bisimulation vs. trace equivalence

Bisimulation vs. trace equivalence

$$\mathcal{T}_1 \sim \mathcal{T}_2 \Longrightarrow \mathit{Traces}(\mathcal{T}_1) = \mathit{Traces}(\mathcal{T}_2)$$

proof: ... path fragment lifting ...

Bisimulation vs. trace equivalence

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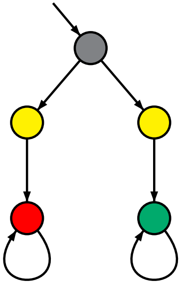
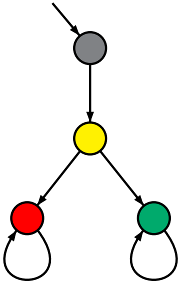
$$\mathit{Traces}(\mathcal{T}_1) = \mathit{Traces}(\mathcal{T}_2) \not\Longrightarrow \mathcal{T}_1 \sim \mathcal{T}_2$$



$\not\sim$



trace equivalent, but not bisimulation equivalent

Bisimulation vs. trace equivalence

$$\mathcal{T}_1 \sim \mathcal{T}_2 \implies \mathit{Traces}(\mathcal{T}_1) = \mathit{Traces}(\mathcal{T}_2)$$

proof: ... path fragment lifting ...

$$\mathit{Traces}(\mathcal{T}_1) = \mathit{Traces}(\mathcal{T}_2) \not\Longrightarrow \mathcal{T}_1 \sim \mathcal{T}_2$$

> Trace equivalence is strictly coarser than bisimulation equivalence.

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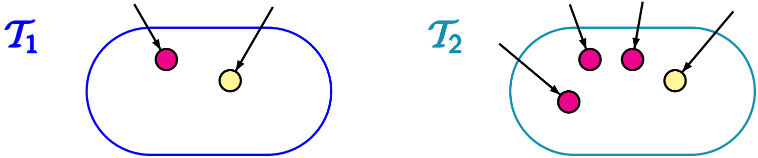
Trace equivalence is strictly coarser than bisimulation equivalence.

Bisimulation equivalent transition systems satisfy the same **LT properties** (e.g., **LTL** formulas).

- as a relation that compares **2** transition systems

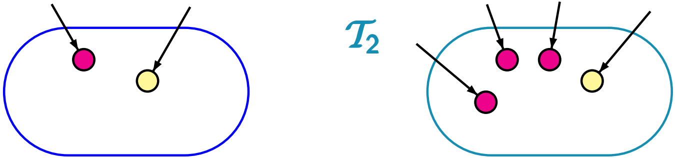


Bisimulation equivalence ...

- as a relation that compares **2** transition systems

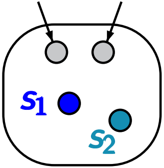
$\mathcal{T}_1$

$\mathcal{T}_2$



- as a relation on the states of **1** transition system

$\mathcal{T}$



$s_1 \sim s_2$ iff $\mathcal{T}_{s_1} \sim \mathcal{T}_{s_2}$

Bisimulation equivalence ...

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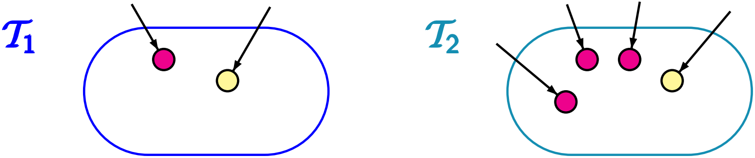

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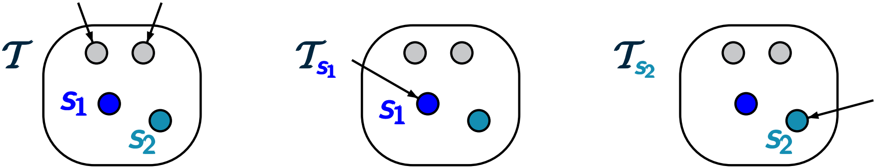

$s_1 \sim s_2$ iff $\mathcal{T}_{s_1} \sim \mathcal{T}_{s_2}$ iff
there exists a bisimulation $\mathcal{R}$ for $\mathcal{T}$ s.t. $(s_1, s_2) \in \mathcal{R}$

Bisimulations on a single TS

Bisimulations on a single TS

Let $\mathcal{T}$ be a TS with proposition set AP.

A bisimulation for $\mathcal{T}$ is a binary relation $\mathcal{R}$ on the state space of $\mathcal{T}$ s.t. for all $(s_1, s_2) \in \mathcal{R}$:

(1) $L(s_1) = L(s_2)$

(2) $\forall\, s_1' \in Post(s_1)\ \exists\, s_2' \in Post(s_2)$ s.t. $(s_1', s_2') \in \mathcal{R}$

(3) $\forall\, s_2' \in Post(s_2)\ \exists\, s_1' \in Post(s_1)$ s.t. $(s_1', s_2') \in \mathcal{R}$

Bisimulation equivalence $\sim_{\mathcal{T}}$ on a single TS

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Let $\mathcal{T}$ be a transition system with state space S.

Bisimulation equivalence $\sim_{\mathcal{T}}$ is

- the coarsest bisimulation on $\mathcal{T}$
- and an equivalence on S

Bisimulation equivalence

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Bisimulation equivalence

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$$\boxed{\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \\ s_1' & & \end{array}} \quad \text{can be completed to} \quad \boxed{\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \downarrow \\ s_1' & \sim_{\mathcal{T}} & s_2' \end{array}}$$

Two variants of bisimulation equivalence

$\sim$ relation that compares **2** transition systems

$\sim_{\mathcal{T}}$ equivalence on the state space of a single TS $\mathcal{T}$

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1. $\sim_{\mathcal{T}}$ can be derived from $\sim$

for all states s_1 and s_2 of $\mathcal{T}$:

$$s_1 \sim_{\mathcal{T}} s_2 \quad \text{iff} \quad \mathcal{T}_{s_1} \sim \mathcal{T}_{s_1}$$

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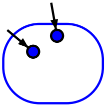
2. $\sim$ can be derived from $\sim_{\mathcal{T}}$

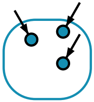
Derivation of $\sim$ from $\sim_{\mathcal{T}}$

given two transition systems $\mathcal{T}_1$ and $\mathcal{T}_2$

$\mathcal{T}_1$ with state space S_1



$\mathcal{T}_2$ with state space S_2



Derivation of $\sim$ from $\sim_{\mathcal{T}}$

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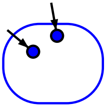
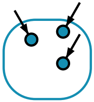
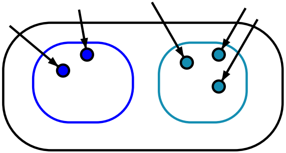


$\mathcal{T}_2$ with state space S_2



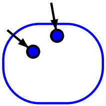
consider $\mathcal{T} = \mathcal{T}_1 \uplus \mathcal{T}_2$
(state space $S_1 \uplus S_2$)



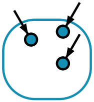
Derivation of $\sim$ from $\sim_{\mathcal{T}}$

given two transition systems $\mathcal{T}_1$ and $\mathcal{T}_2$

$\mathcal{T}_1$ with state space S_1



$\mathcal{T}_2$ with state space S_2



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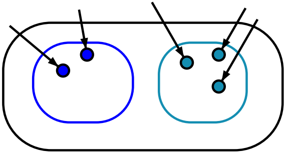

$\mathcal{T}_1 \sim \mathcal{T}_2$ iff $\forall$ initial states s_1 of $\mathcal{T}_1$
$\exists$ initial state s_2 of $\mathcal{T}_2$ s.t. $s_1 \sim_{\mathcal{T}} s_2$,
and vice versa

Bisimulation quotient

BSEQOR5.1-35

Bisimulation quotient

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a TS.

bisimulation quotient $\mathcal{T}/\sim$ arises from $\mathcal{T}$
by collapsing bisimulation equivalent states

Bisimulation quotient

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bisimulation quotient:

$$\mathcal{T}/\!\sim\; = (S', Act', \rightarrow', S_0', AP, L')$$

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- state space: $S' = S/\!\sim_{\mathcal{T}}$

set of bisimulation equivalence classes

Bisimulation quotient

Let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a TS.

bisimulation quotient:

$$\mathcal{T}/\!\sim\, = (S', Act', \rightarrow', S'_0, AP, L')$$

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- labeling function: $L'([s]_{\sim_{\mathcal{T}}}) = L(s)$

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well-defined

by the labeling condition of bisimulations

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- transition relation:

$$\frac{s \longrightarrow s'}{[s]_{\sim_{\mathcal{T}}} \longrightarrow [s']_{\sim_{\mathcal{T}}}}$$

action labels irrelevant

Let $\mathcal{T} = (S, Act, \to, S_0, AP, L)$ be a TS.

bisimulation quotient:

$$\mathcal{T}/\!\sim\; = (S', \{\tau\}, \to', S'_0, AP, L')$$

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$$\boxed{\mathcal{T} \sim \mathcal{T}/\!\sim}$$

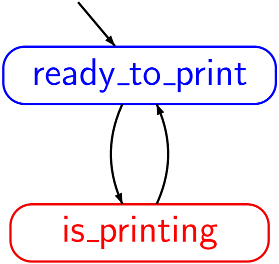
Example: interleaving of n printers

parallel system $\mathcal{T} = \underbrace{\mathit{Printer} \,|||\, \mathit{Printer} \,|||\, \ldots \,|||\, \mathit{Printer}}_{\boldsymbol{n} \text{ printer}}$

Example: interleaving of *n* printers

parallel system $\mathcal{T} = \underbrace{\textit{Printer} \,|||\, \textit{Printer} \,|||\, \ldots \,|||\, \textit{Printer}}_{n \text{ printer}}$

transition system
for each printer

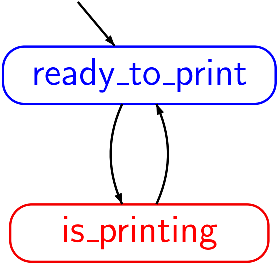


Example: interleaving of n printers

parallel system $\mathcal{T} = \underbrace{\mathit{Printer} \,|||\, \mathit{Printer} \,|||\, \ldots \,|||\, \mathit{Printer}}_{n \text{ printer}}$

$AP = \{0, 1, \ldots, n\}$ "number of available printers"

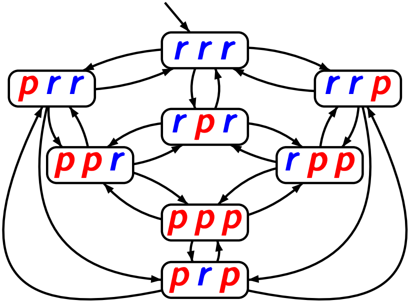
transition system
for each printer



Example: n=3 printers

parallel system $\mathcal{T} = \underbrace{\textit{Printer} \mid\mid\mid \textit{Printer} \mid\mid\mid \ldots \mid\mid\mid \textit{Printer}}_{n \text{ printer}}$

$AP = \{0, 1, 2, 3\}$



p: is printing

r: ready to print

parallel system $\mathcal{T} = \underbrace{\textit{Printer} \;|||\; \textit{Printer} \;|||\; \ldots \;|||\; \textit{Printer}}_{n \text{ printer}}$

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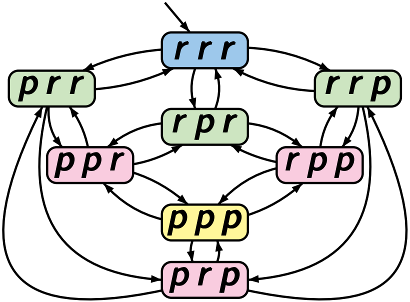

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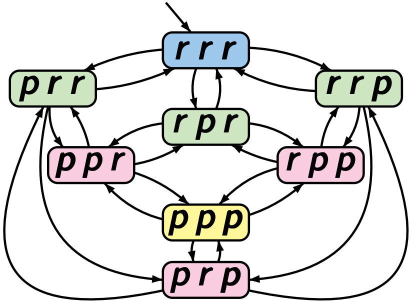

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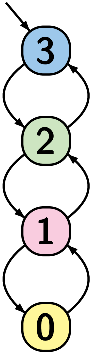

bisimulation quotient

Example: n=3 printers

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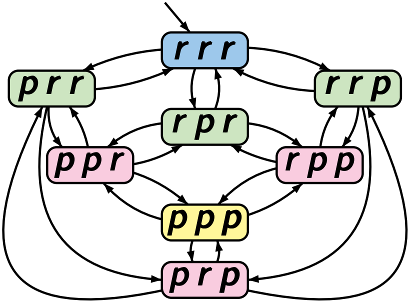

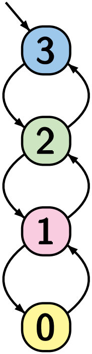

2^n states

$n+1$ states

Mutual exclusion

Mutual exclusion: Bakery algorithm

solutions for mutual exclusion problems:

- semaphore
- Peterson's algorithm
- Bakery algorithm

Mutual exclusion: Bakery algorithm

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- semaphore
- Peterson's algorithm
- Bakery algorithm

given two concurrent processes P_1 and P_2

- two additional shared variables: $x_1, x_2 \in \mathbb{N}$
- if P_1 and P_2 are waiting then:

  if $x_1 < x_2$ then P_1 enters its critical section

  if $x_2 < x_1$ then P_2 enters its critical section

  $x_1 = x_2$: cannot happen

Bakery algorithm

protocol for P_1:

```
LOOP FOREVER
    noncritical actions
    x1 := x2+1
    AWAIT (x1 < x2) ∨ (x2=0);
    critical section;
    x1 := 0
END LOOP
```

symmetric protocol for P_2

Bakery algorithm

protocol for P_1:

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    noncritical actions
    x1 := x2+1
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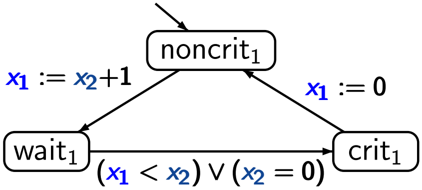
END LOOP
```

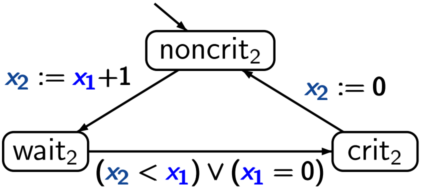
initially:

$x_1 = x_2 = 0$

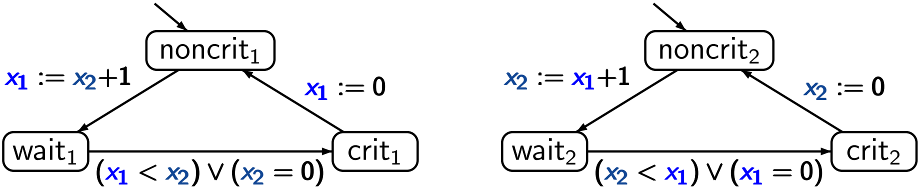
symmetric protocol for P_2

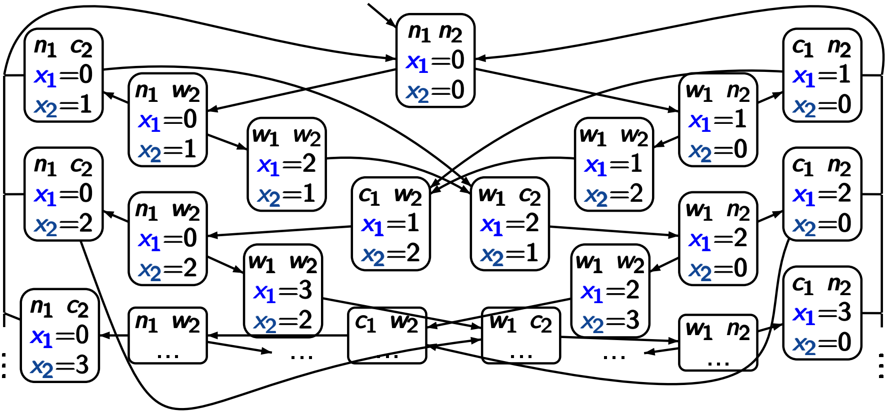
Program graphs for the Bakery algorithm



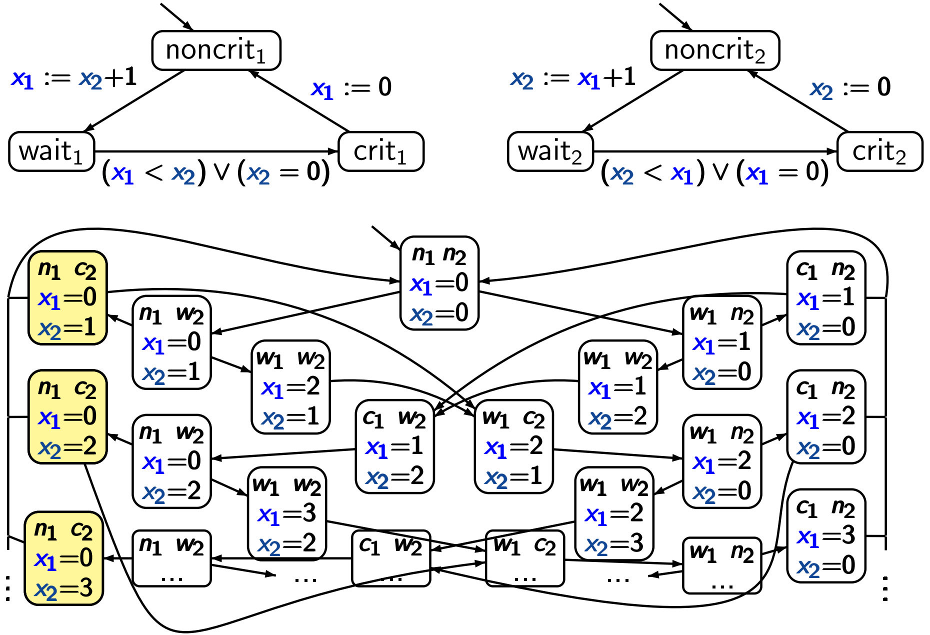


Transition system for the Bakery algorithm

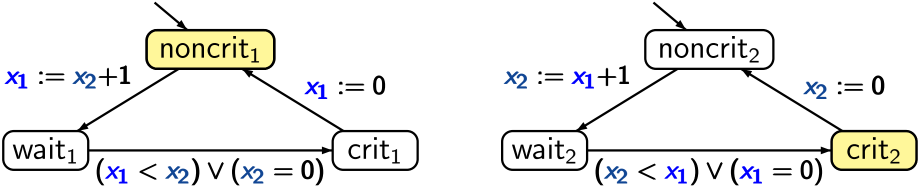


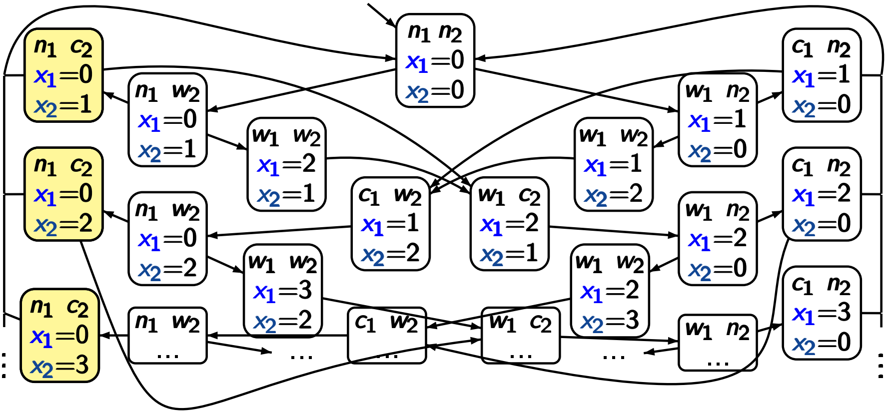


Transition system for the Bakery algorithm



Transition system for the Bakery algorithm





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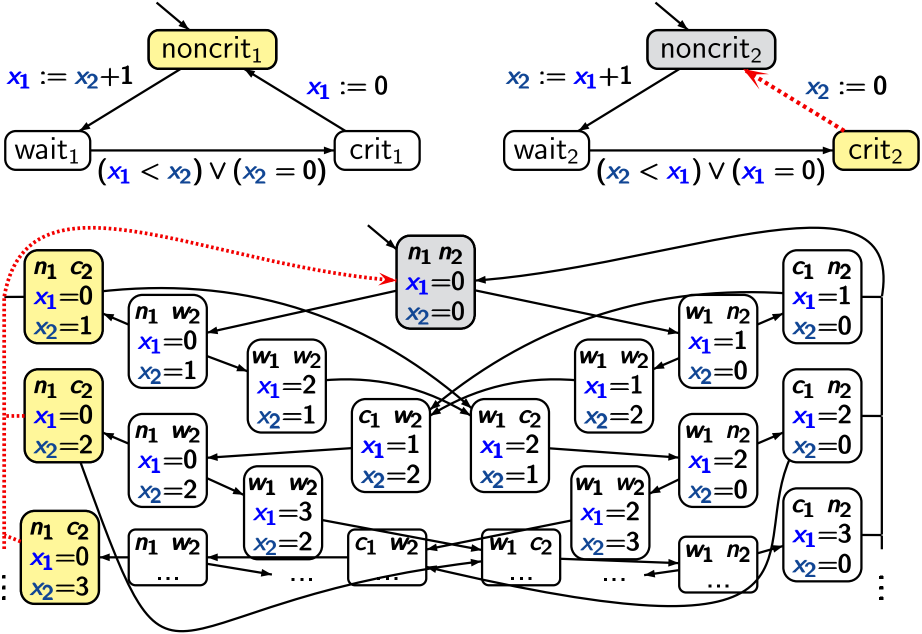
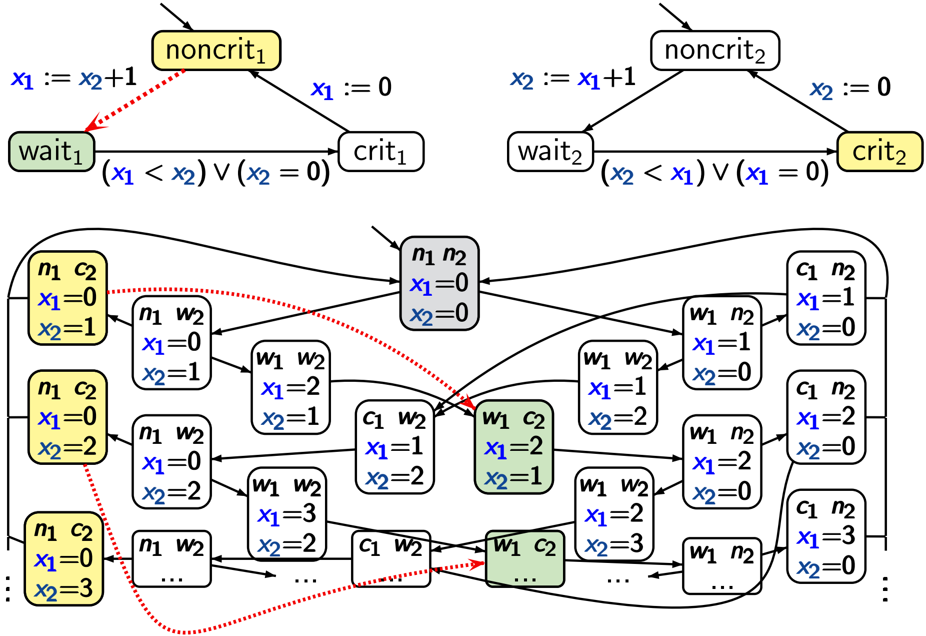


Transition system for the Bakery algorithm



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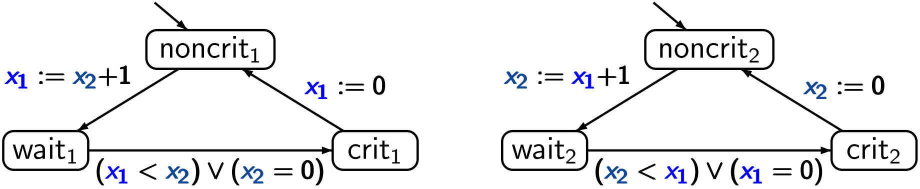
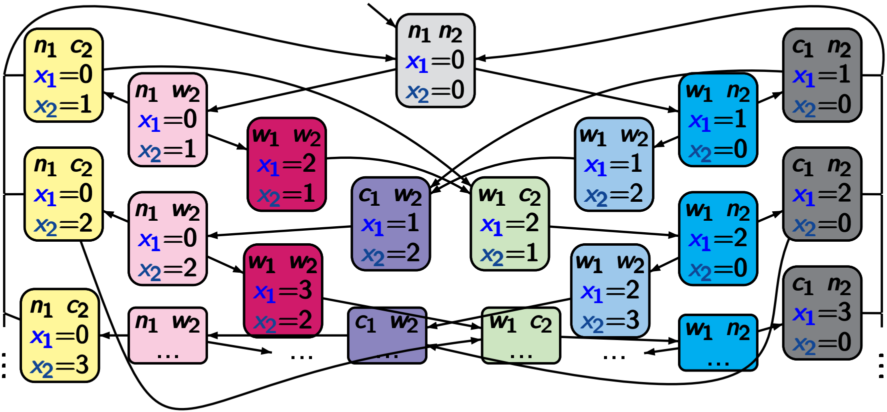
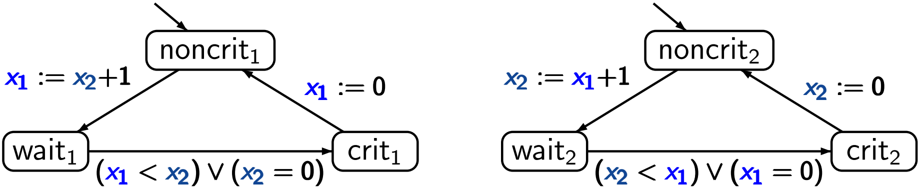


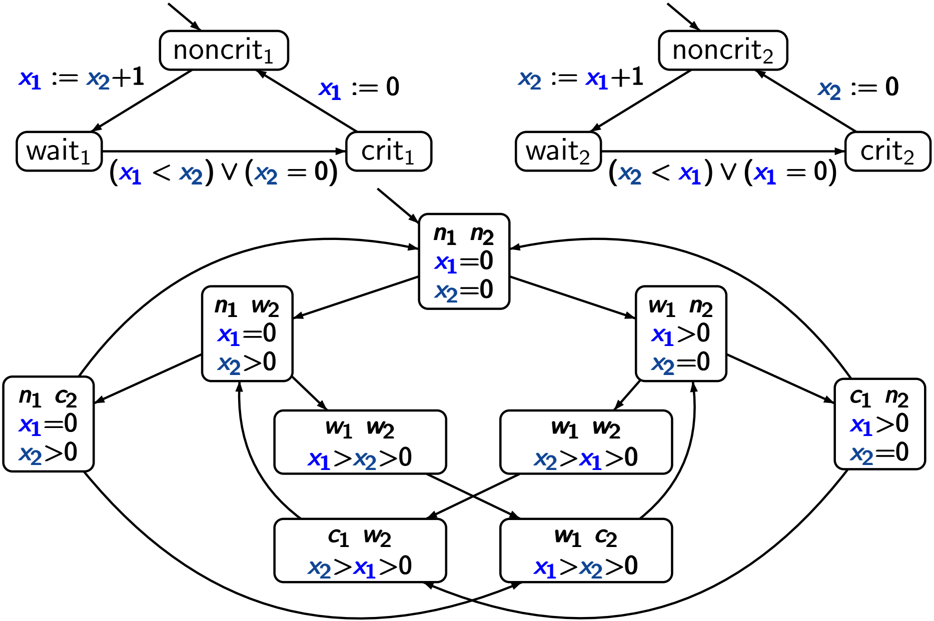


Bakery algorithm: bisimulation quotient

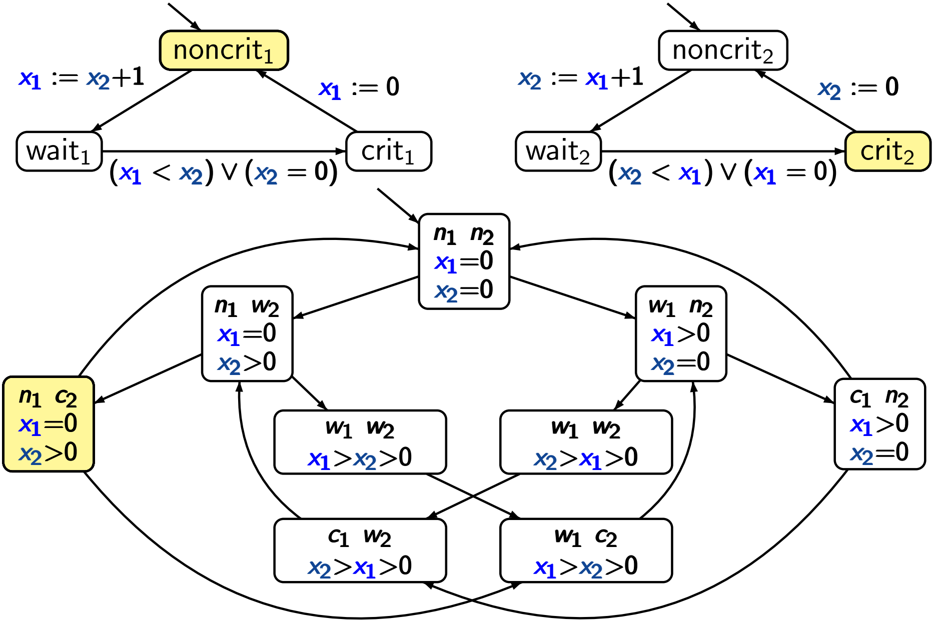


infinite transition system with a
finite bisimulation quotient

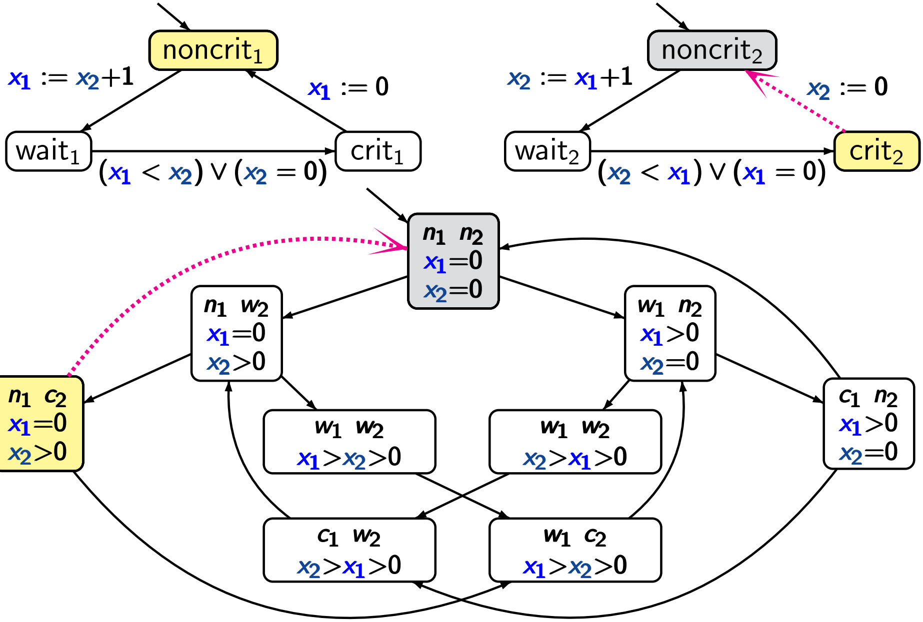
Bakery algorithm: bisimulation quotient



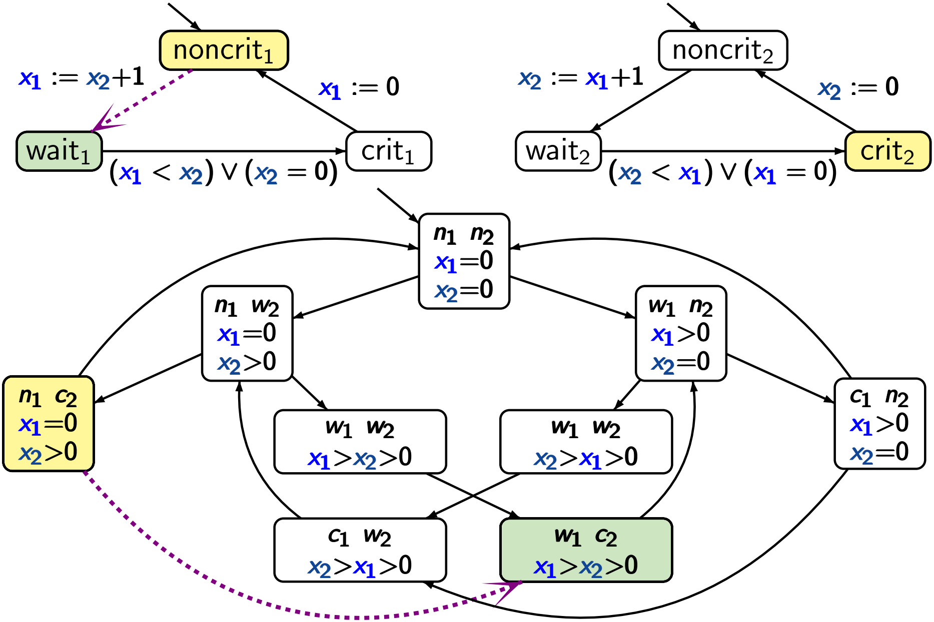
Bakery algorithm: bisimulation quotient



Bakery algorithm: bisimulation quotient



Bakery algorithm: bisimulation quotient



Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

- bisimulation
- CTL, CTL*-equivalence $\longleftarrow$
- computing the bisimulation quotient
- abstraction stutter steps
- simulation relations

CTL* state formulas

$$\Phi \ ::= \ \mathit{true} \ \big| \ a \ \big| \ \Phi_1 \wedge \Phi_2 \ \big| \ \neg\Phi \ \big| \ \exists\varphi$$

CTL* path formulas

$$\varphi \ ::= \ \Phi \ \big| \ \varphi_1 \wedge \varphi_2 \ \big| \ \neg\varphi \ \big| \ \bigcirc\varphi \ \big| \ \varphi_1 \, \mathbf{U} \, \varphi_2$$

Recall: CTL* and CTL

CTL* state formulas

$$\Phi ::= \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi$$

CTL* path formulas

$$\varphi ::= \Phi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \,\mathsf{U}\, \varphi_2$$

CTL: sublogic of **CTL***

- with path quantifiers $\exists$ and $\forall$
- restricted syntax of path formulas:
  - *no* boolean combinations of path formulas
  - arguments of temporal operators $\bigcirc$ and **U** are state formulas

CTL equivalence

CTLEQ5.2-1

CTL equivalence

Let s_1, s_2 be states of a TS $\mathcal{T}$ without terminal states

CTL equivalence

Let s_1, s_2 be states of a TS $\mathcal{T}$ without terminal states

s_1, s_2 are **CTL** equivalent if for all **CTL** formulas Φ:

$$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

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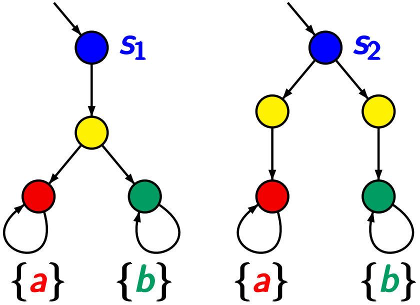

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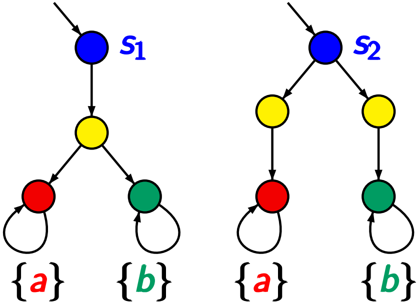

s_1, s_2 are
not **CTL** equivalent

$s_1 \models \exists\bigcirc(\exists\bigcirc a \wedge \exists\bigcirc b)$

$s_2 \not\models \exists\bigcirc(\exists\bigcirc a \wedge \exists\bigcirc b)$

CTL, CTL* and LTL equivalence

Let s_1, s_2 be states of a TS $\mathcal{T}$ without terminal states

s_1, s_2 are **CTL** equivalent if for all **CTL** formulas Φ:

$$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

analogous definition for **CTL*** and **LTL**

CTL, CTL* and LTL equivalence

Let s_1, s_2 be states of a TS $\mathcal{T}$ without terminal states

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s_1, s_2 are **CTL*** equivalent if for all **CTL*** formulas Φ:

$$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

s_1, s_2 are **LTL** equivalent if for all **LTL** formulas φ:

$$s_1 \models \varphi \quad \text{iff} \quad s_2 \models \varphi$$

CTL/CTL* and bisimulation

CTL/CTL* and bisimulation

bisimulation equivalence

= **CTL** equivalence

= **CTL*** equivalence

CTL/CTL* and bisimulation

	bisimulation equivalence	
=	**CTL** equivalence	$\longleftarrow$ for finite TS
=	**CTL*** equivalence	

CTL/CTL* and bisimulation

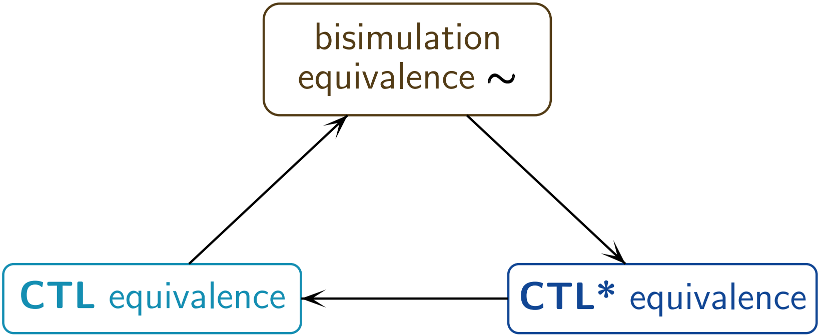
$$\begin{array}{ll} & \text{bisimulation equivalence} \\ = & \textbf{CTL} \ \ \text{equivalence} \\ = & \textbf{CTL*} \ \text{equivalence} \end{array} \quad \longleftarrow \text{ for finite TS}$$

Let $\mathcal{T}$ be a finite TS without terminal states, and s_1, s_2 states in $\mathcal{T}$. Then:

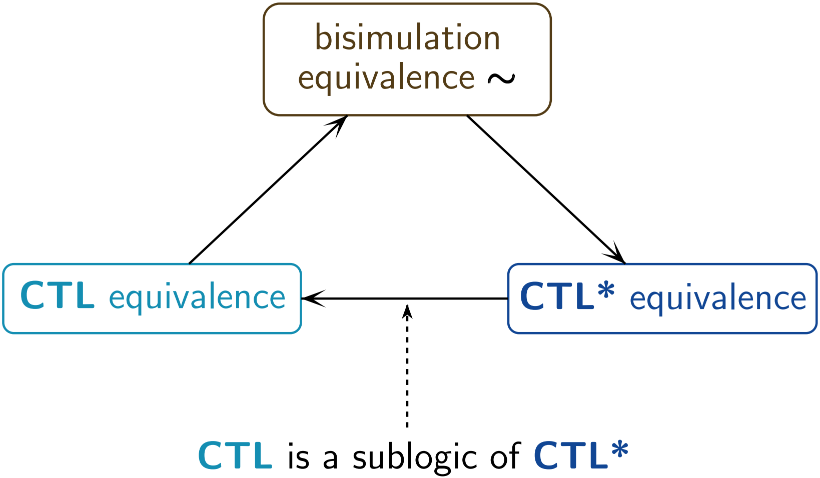
$$s_1 \sim_{\mathcal{T}} s_2$$

iff s_1 and s_2 are **CTL** equivalent

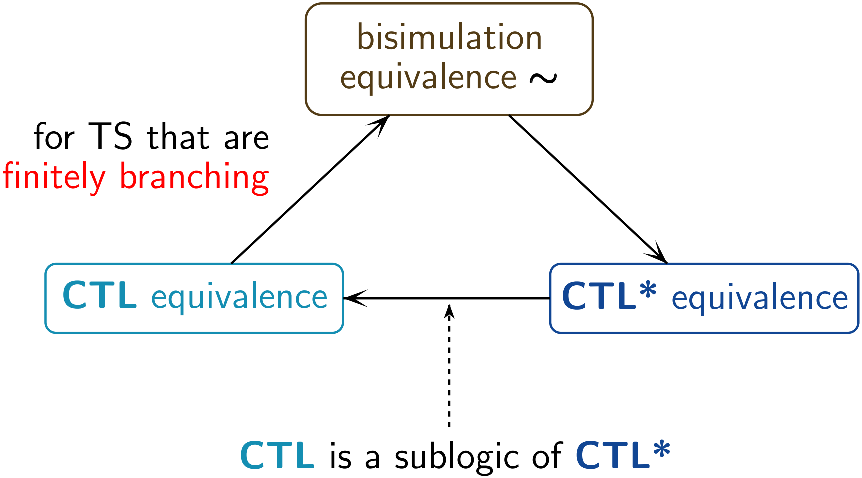
iff s_1 and s_2 are **CTL*** equivalent


bisimulation
equivalence $\sim$
CTL equivalence
CTL* equivalence

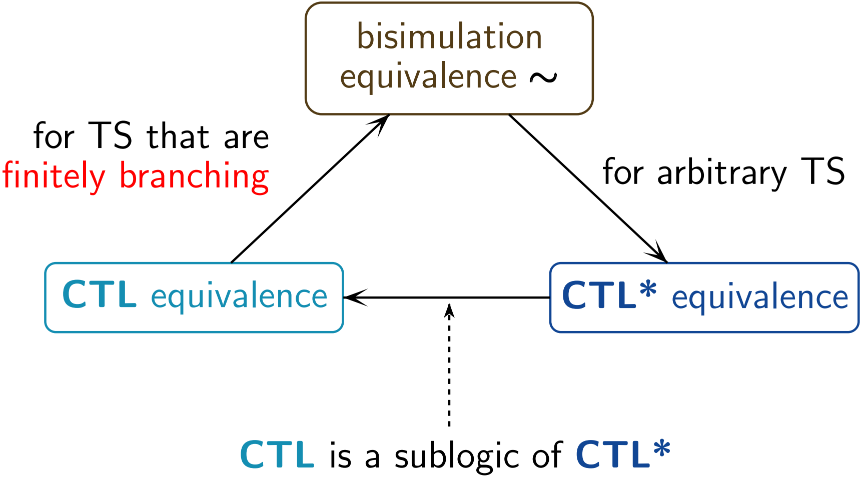
CTL/CTL* and bisimulation



CTL/CTL* and bisimulation



CTL/CTL* and bisimulation



Bisimulation equivalence $\Rightarrow$ CTL* equivalence

For arbitrary (possibly infinite) transition systems without terminal states:

Bisimulation equivalence ⇒ CTL* equivalence

For arbitrary (possibly infinite) transition systems without terminal states:

> If s_1, s_2 are states with $s_1 \sim_{\mathcal{T}} s_2$ then for all **CTL*** formulas Φ:
>
> $$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

Bisimulation equivalence ⇒ CTL* equivalence

show by structural induction on **CTL*** formulas:

(a) if s_1, s_2 are states with $s_1 \sim_{\mathcal{T}} s_2$ then for all **CTL*** state formulas Φ:

$$s_1 \models \Phi \quad \text{iff} \quad s_2 \models \Phi$$

(b) if π_1, π_2 are paths with $\pi_1 \sim_{\mathcal{T}} \pi_2$ then for all **CTL*** path formulas φ:

$$\pi_1 \models \varphi \quad \text{iff} \quad \pi_2 \models \varphi$$

show by structural induction on **CTL*** formulas:

(a) if s_1, s_2 are states with $s_1 \sim_{\mathcal{T}} s_2$ then for all **CTL*** state formulas Φ:

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$$\pi_1 \models \varphi \quad \text{iff} \quad \pi_2 \models \varphi$$

$\pi_1 \sim_{\mathcal{T}} \pi_2 \quad \overset{\text{def}}{\iff} \quad \pi_1$ and π_2 are statewise bisimulation equivalent

Bisimulation equivalence ⇒ CTL* equivalence

statewise bisimulation equivalent paths:

$$\begin{array}{ccc}
s_1 & \sim_{\mathcal{T}} & s_2 \\
\downarrow & & \downarrow \\
s_{11} & \sim_{\mathcal{T}} & s_{12} \\
\downarrow & & \downarrow \\
s_{21} & \sim_{\mathcal{T}} & s_{22} \\
\downarrow & & \downarrow \\
s_{31} & \sim_{\mathcal{T}} & s_{32} \\
\downarrow & & \downarrow \\
\ldots & & \ldots \\
\uparrow & & \uparrow \\
\boxed{\text{path } \pi_1} & & \boxed{\text{path } \pi_2}
\end{array}$$

Bisimulation equivalence $\Rightarrow$ CTL* equivalence

For all CTL* state formulas Φ and path formulas φ:

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \Phi$ iff $s_2 \models \Phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Bisimulation equivalence $\Rightarrow$ CTL* equivalence

For all CTL* state formulas Φ and path formulas φ:

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \Phi$ iff $s_2 \models \Phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Proof by structural induction

Bisimulation equivalence $\Rightarrow$ CTL* equivalence

For all CTL* state formulas Φ and path formulas φ:

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(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Proof by structural induction

base of induction:

(a) $\Phi = \mathit{true}$ or $\Phi = a \in AP$

(b) $\varphi = \Phi$ for some state formula Φ
s.t. statement (a) holds for Φ

Bisimulation equivalence $\Rightarrow$ CTL* equivalence

For all CTL* state formulas Φ and path formulas φ:

(a) if $s_1 \sim_{\mathcal{T}} s_2$ then: $s_1 \models \Phi$ iff $s_2 \models \Phi$

(b) if $\pi_1 \sim_{\mathcal{T}} \pi_2$ then: $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

Proof by structural induction

step of induction:

(a) consider $\Phi = \Phi_1 \wedge \Phi_2$, $\neg\Psi$ or $\exists\varphi$ s.t.

- (a) holds for Φ_1, Φ_2, Ψ
- (b) holds for φ

(b) consider $\varphi = \varphi_1 \wedge \varphi_2$, $\neg\varphi'$, $\bigcirc\varphi'$, $\varphi_1 \mathsf{U} \varphi_2$ s.t.

- (a) holds for $\varphi_1, \varphi_2, \varphi'$

Path lifting for $\sim_{\mathcal{T}}$

$$\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \\ s_{11} & & \\ \downarrow & & \\ s_{21} & & \\ \downarrow & & \\ s_{31} & & \\ \downarrow & & \\ \vdots & & \end{array}$$

can be completed to

$$\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \downarrow \\ s_{11} & \sim_{\mathcal{T}} & s_{12} \\ \downarrow & & \downarrow \\ s_{21} & \sim_{\mathcal{T}} & s_{22} \\ \downarrow & & \downarrow \\ s_{31} & \sim_{\mathcal{T}} & s_{32} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

Path lifting for $\sim_{\mathcal{T}}$

$$
\begin{array}{ccc}
s_1 & \sim_{\mathcal{T}} & s_2 \\
\downarrow & & \\
s_{11} & & \\
\downarrow & & \\
s_{21} & & \\
\downarrow & & \\
s_{31} & & \\
\downarrow & & \\
\vdots & &
\end{array}
\qquad \text{can be completed to} \qquad
\begin{array}{ccc}
s_1 & \sim_{\mathcal{T}} & s_2 \\
\downarrow & & \downarrow \\
s_{11} & \sim_{\mathcal{T}} & s_{12} \\
\downarrow & & \downarrow \\
s_{21} & \sim_{\mathcal{T}} & s_{22} \\
\downarrow & & \downarrow \\
s_{31} & \sim_{\mathcal{T}} & s_{32} \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

If $s_1 \sim_{\mathcal{T}} s_2$ then for all $\pi_1 \in \mathit{Paths}(s_1)$ there exists $\pi_2 \in \mathit{Paths}(s_2)$ with $\pi_1 \sim_{\mathcal{T}} \pi_2$

Path lifting for $\sim_{\mathcal{T}}$

$$\begin{array}{ccc}
s_1 & \sim_{\mathcal{T}} & s_2 \\
\downarrow & & \\
s_{11} & & \\
\downarrow & & \\
s_{21} & & \\
\downarrow & & \\
s_{31} & & \\
\downarrow & & \\
\vdots & &
\end{array}$$

path π_1

can be completed to

$$\begin{array}{ccc}
s_1 & \sim_{\mathcal{T}} & s_2 \\
\downarrow & & \downarrow \\
s_{11} & \sim_{\mathcal{T}} & s_{12} \\
\downarrow & & \downarrow \\
s_{21} & \sim_{\mathcal{T}} & s_{22} \\
\downarrow & & \downarrow \\
s_{31} & \sim_{\mathcal{T}} & s_{32} \\
\downarrow & & \downarrow \\
\vdots & & \vdots
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If $s_1 \sim_{\mathcal{T}} s_2$ then for all $\pi_1 \in Paths(s_1)$ there exists $\pi_2 \in Paths(s_2)$ with $\pi_1 \sim_{\mathcal{T}} \pi_2$

Path lifting for $\sim_{\mathcal{T}}$

$$\begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \\ s_{11} & & \\ \downarrow & & \\ s_{21} & & \\ \downarrow & & \\ s_{31} & & \\ \downarrow & & \\ \vdots & & \\ \boxed{\text{path } \pi_1} & & \end{array} \qquad \text{can be completed to} \qquad \begin{array}{ccc} s_1 & \sim_{\mathcal{T}} & s_2 \\ \downarrow & & \downarrow \\ s_{11} & \sim_{\mathcal{T}} & s_{12} \\ \downarrow & & \downarrow \\ s_{21} & \sim_{\mathcal{T}} & s_{22} \\ \downarrow & & \downarrow \\ s_{31} & \sim_{\mathcal{T}} & s_{32} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ & & \boxed{\text{path } \pi_2} \end{array}$$

If $s_1 \sim_{\mathcal{T}} s_2$ then for all $\pi_1 \in \mathit{Paths}(s_1)$
there exists $\pi_2 \in \mathit{Paths}(s_2)$ with $\pi_1 \sim_{\mathcal{T}} \pi_2$

Correct or wrong?

If s_1, s_2 are not **CTL** equivalent then there exists a **CTL** formula Φ with $s_1 \models \Phi$ and $s_2 \not\models \Phi$

Correct or wrong?

If s_1, s_2 are <u>not</u> **CTL** equivalent then there exists a **CTL** formula Φ with $s_1 \models \Phi$ and $s_2 \not\models \Phi$

correct.

Correct or wrong?

If s_1, s_2 are not **CTL** equivalent then there exists a **CTL** formula Φ with $s_1 \models \Phi$ and $s_2 \not\models \Phi$

correct.

If s_1, s_2 not **CTL** equivalent then
there exists a **CTL** formula Φ with

$$s_1 \models \Phi \wedge s_2 \not\models \Phi$$

or $$s_1 \not\models \Phi \wedge s_2 \models \Phi$$

If s_1, s_2 are <u>not</u> **CTL** equivalent then there exists a **CTL** formula Φ with $s_1 \models \Phi$ and $s_2 \not\models \Phi$

correct.

If s_1, s_2 not **CTL** equivalent then
there exists a **CTL** formula Φ with

$$s_1 \models \Phi \wedge s_2 \not\models \Phi$$

$$\text{or} \quad s_1 \not\models \Phi \wedge s_2 \models \Phi \quad \Longrightarrow \quad s_1 \models \neg\Phi \wedge s_2 \not\models \neg\Phi$$

CTL equivalence $\Longrightarrow$ bisimulation equivalence

CTL equivalence $\Longrightarrow$ bisimulation equivalence

If $\mathcal{T}$ is a finite TS then, for all states s_1, s_2 in $\mathcal{T}$:

if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

CTL equivalence $\Longrightarrow$ bisimulation equivalence

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Proof: show that

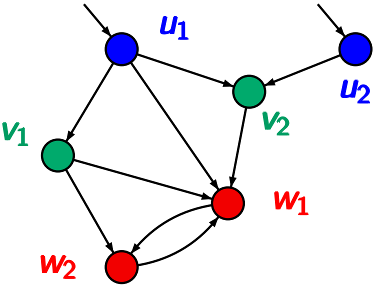
$$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1,\ s_2 \text{ satisfy the same } \textbf{CTL} \text{ formulas} \}$$

is a bisimulation, i.e., for all $(s_1, s_2) \in \mathcal{R}$:

(1) $L(s_1) = L(s_2)$

(2) if $s_1 \longrightarrow t_1$ then there exists a transition $s_2 \longrightarrow t_2$ s.t. $(t_1, t_2) \in \mathcal{R}$

Example: CTL master formulas

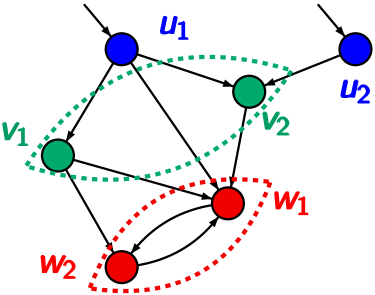


$\widehat{=} \{a\}$

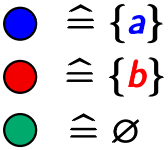
$\widehat{=} \{b\}$

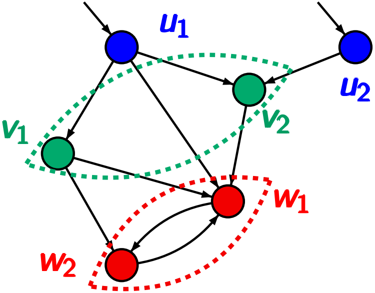
$\widehat{=} \varnothing$

Example: CTL master formulas



bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$





bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$

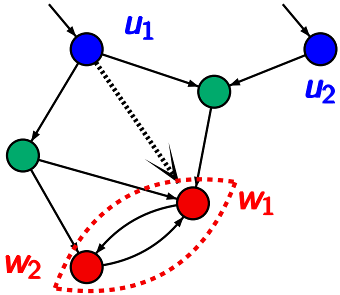
but $u_1 \not\sim_{\mathcal{T}} u_2$

$\widehat{=} \; \{a\}$

$\widehat{=} \; \{b\}$

$\widehat{=} \; \varnothing$

Example: CTL master formulas



bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$

but $u_1 \not\sim_{\mathcal{T}} u_2$

as $u_1 \to \{w_1, w_2\}$

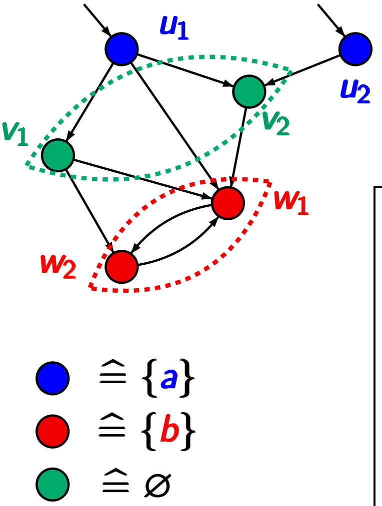
$u_2 \not\to \{w_1, w_2\}$

$\widehat{=} \{a\}$

$\widehat{=} \{b\}$

$\widehat{=} \varnothing$

Example: CTL master formulas



bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$

CTL master formulas:

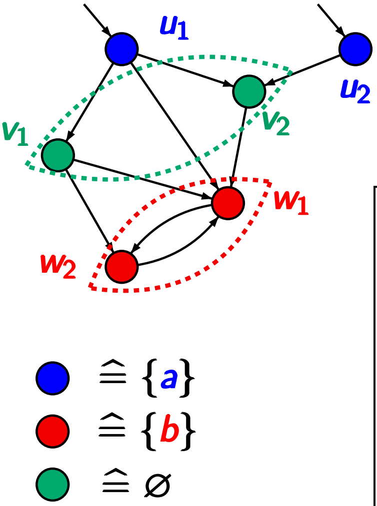
$w_1, w_2 \models ?$

$v_1, v_2 \models ?$

$u_1 \models ?$

$u_2 \models ?$

Example: CTL master formulas



bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$

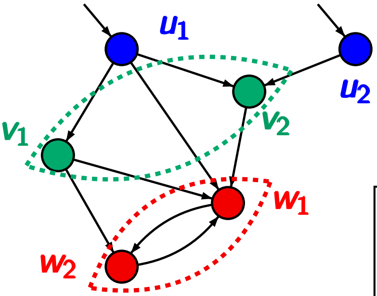
CTL master formulas:

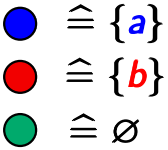
$w_1, w_2 \models b$

$v_1, v_2 \models ?$

$u_1 \models ?$

$u_2 \models ?$





bisimulation equivalence $\sim_{\mathcal{T}}$
$= \{(v_1, v_2), (w_1, w_2), \ldots\}$

CTL master formulas:

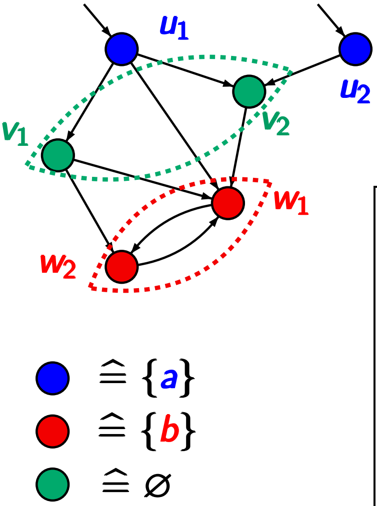
$w_1, w_2 \models b$

$v_1, v_2 \models \neg a \wedge \neg b$

$u_1 \models$?

$u_2 \models$?

Example: CTL master formulas



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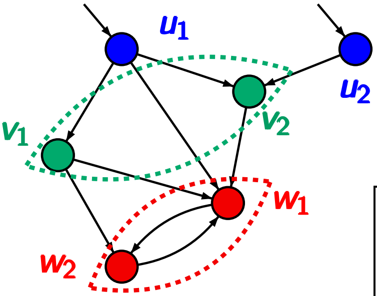

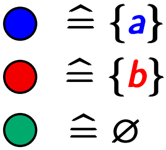

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CTL master formulas:

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w_1, w_2 & \models & b \\
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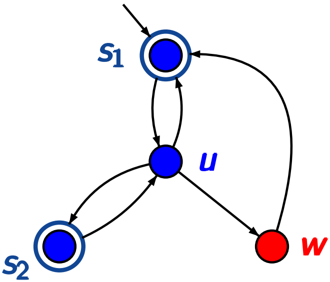
\end{array}
$$

...master formulas for $\sim_{\mathcal{T}}$-classes?



$AP = \{blue, red\}$

$s_1 \sim_{\mathcal{T}} s_2 \not\sim_{\mathcal{T}} u$

$$\Phi_w = red$$

$$\Phi_C = blue \wedge \forall \bigcirc blue \quad \text{where } C = \{s_1, s_2\}$$

$$\Phi_u = \exists \bigcirc red$$

CTL equivalence $\Longrightarrow$ bisimulation equivalence

If $\mathcal{T}$ is a finite TS then, for all states s_1, s_2 in $\mathcal{T}$:

if s_1, s_2 are **CTL** equivalent then $s_1 \sim_{\mathcal{T}} s_2$

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- wrong for infinite TS
- but also holds for finitely branching TS

$\uparrow$

possibly infinite-state TS such that

* the number of initial states is finite
* for each state the number of successors is finite

CTL equivalence $\Longrightarrow$ bisimulation equivalence

Let $\mathcal{T} = (S, Act, \to, S_0, AP, L)$ be finitely branching.

- S_0 is finite
- $Post(s)$ is finite for all $s \in S$

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Proof: as for finite TS. Amounts showing that

$$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1, s_2 \text{ satisfy the same } \textbf{CTL} \text{ formulas} \}$$

is a bisimulation.

CTL equivalence $\Longrightarrow$ bisimulation equivalence

If $\mathcal{T}$ is a finitely branching TS then for all states s_1, s_2:

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Proof: show that

$$\mathcal{R} \stackrel{\text{def}}{=} \{ (s_1, s_2) : s_1, s_2 \text{ satisfy the same } \textbf{CTL} \text{ formulas} \}$$

is a bisimulation, i.e., for $(s_1, s_2) \in \mathcal{R}$:

(1) $L(s_1) = L(s_2)$

(2) if $s_1 \rightarrow t_1$ then there exists a transition $s_2 \rightarrow t_2$ s.t. $(t_1, t_2) \in \mathcal{R}$

Summary: CTL/CTL* and bisimulation

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Let $\mathcal{T}$ be a finite TS without terminal states,
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\begin{array}{ll}
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\text{iff} & s_1 \text{ and } s_2 \text{ are } \textbf{CTL} \text{ equivalent} \\
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\end{array}
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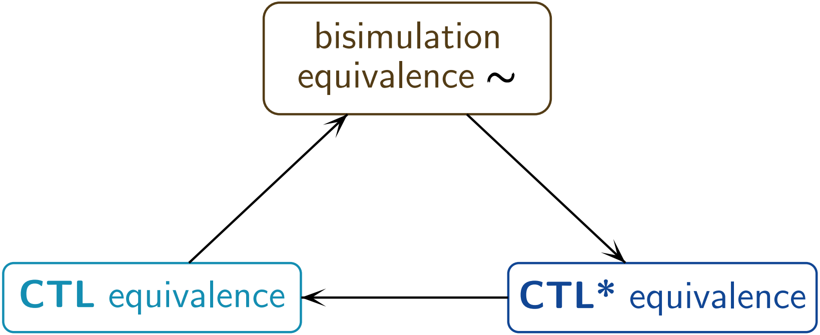

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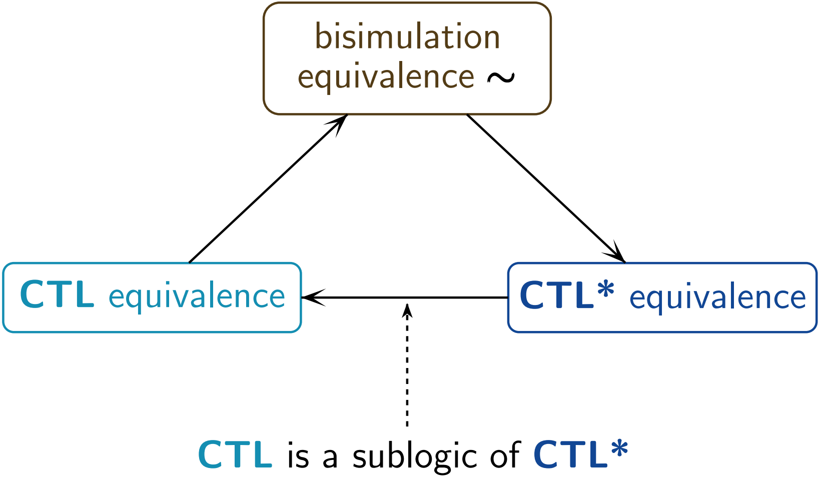

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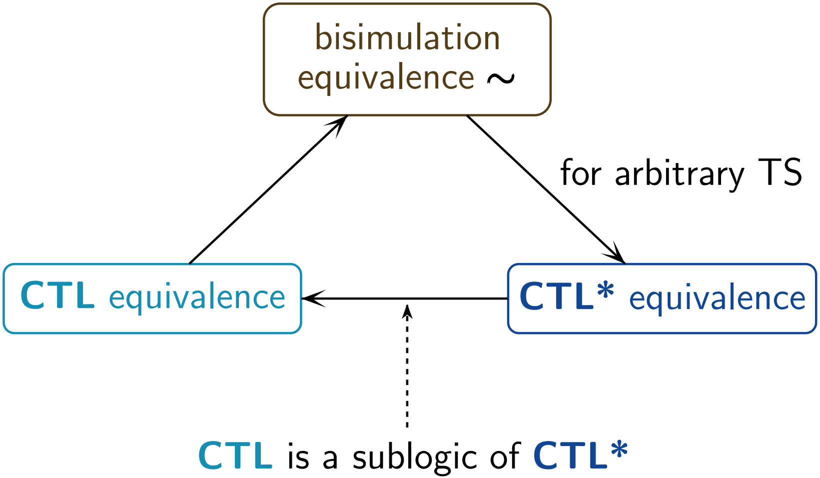

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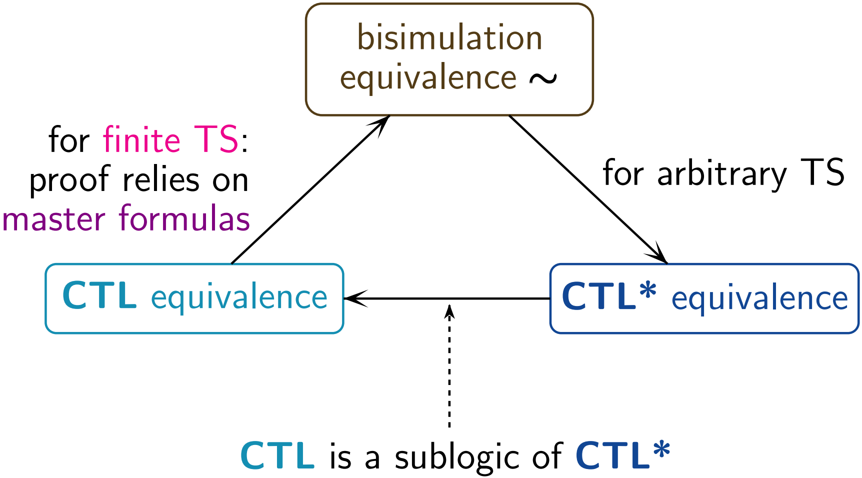

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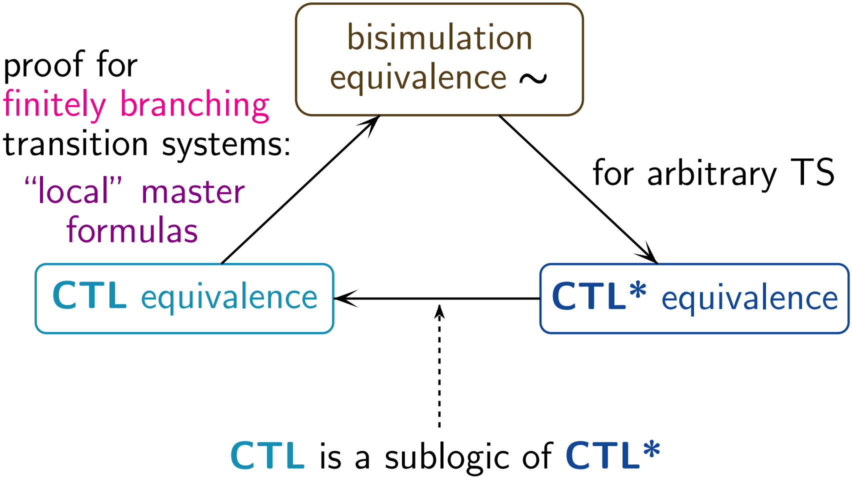

CTL/CTL* and bisimulation for TS

so far: we considered

- **CTL**/**CTL*** equivalence
- bisimulation equivalence $\sim_{\mathcal{T}}$

for the states of a single transition system $\mathcal{T}$

CTL/CTL* and bisimulation for TS

If $\mathcal{T}_1$, $\mathcal{T}_2$ are finitely branching TS over AP without terminal states then:

$$\mathcal{T}_1 \sim \mathcal{T}_2$$

iff $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the same **CTL** formulas

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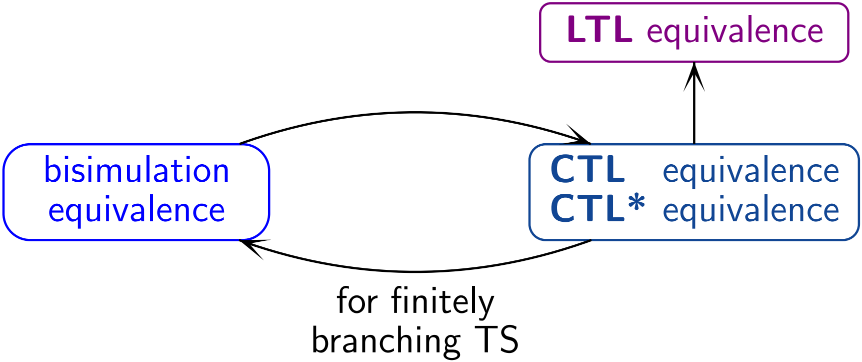
Summary: equivalences



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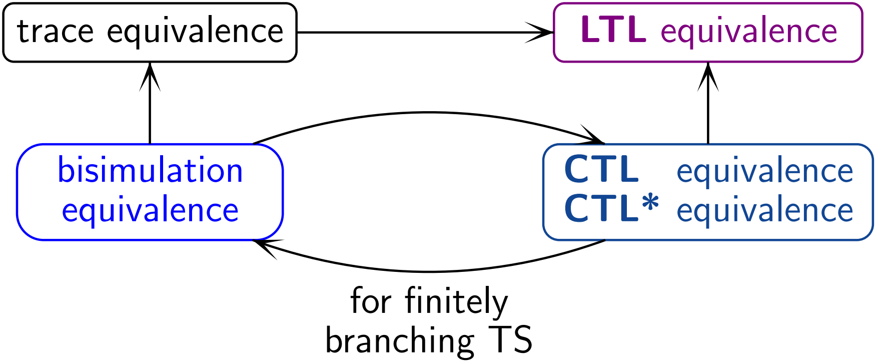

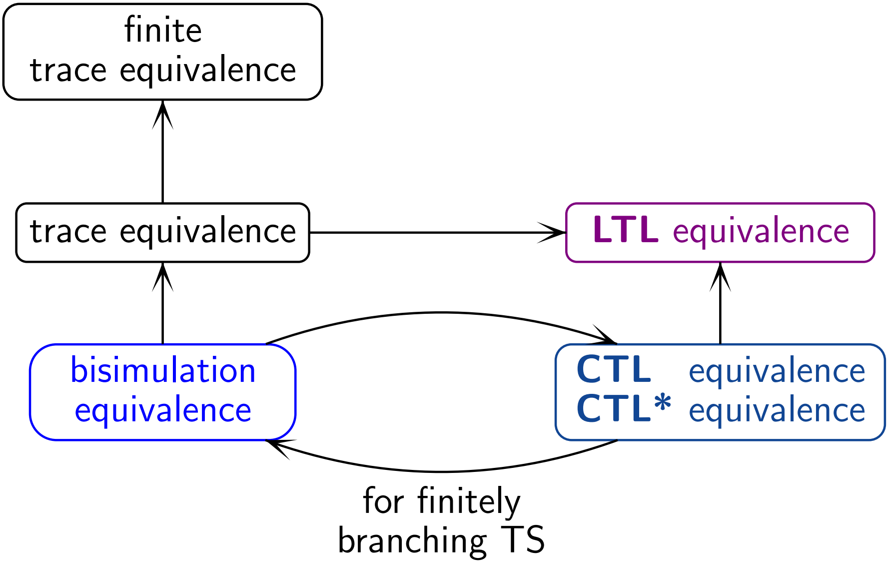
finite
trace equivalence
trace equivalence
LTL equivalence
bisimulation
equivalence
CTL equivalence
CTL* equivalence
for finitely
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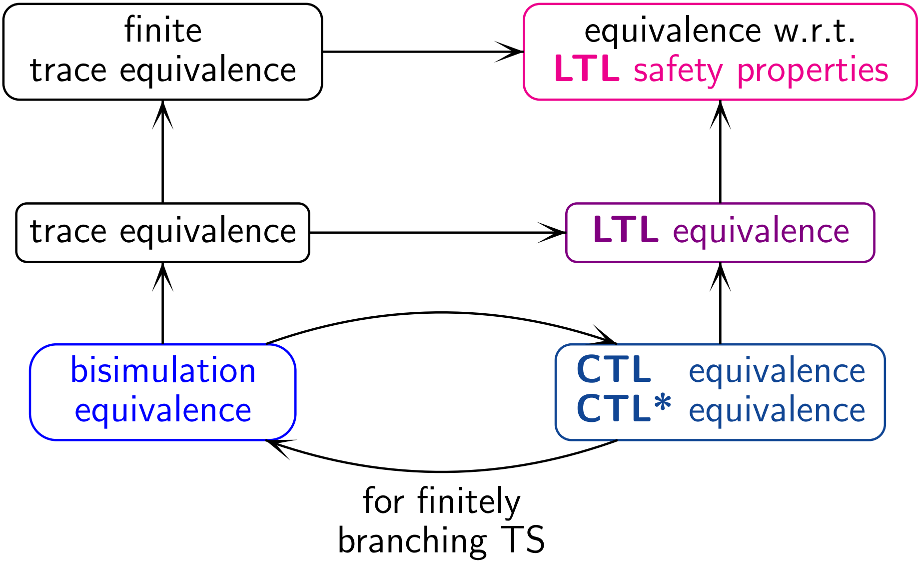
Summary: equivalences



Let $\mathcal{T}$ be a finite TS without terminal states and s_1, s_2 states of $\mathcal{T}$.

> If s_1, s_2 satisfy the same $\mathrm{CTL}_{\setminus \mathrm{U}}$ formulas then
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correct. see the proof

"**CTL** equivalence $\Longrightarrow$ bisimulation equivalence"

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Proof. Show that $\mathsf{CTL}_{\setminus \mathsf{U}}$ equivalence is a bisimulation

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$$\text{and} \quad Sat(\Phi_{C,D}) \subseteq C \setminus D$$