Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL
- automata-based LTL model checking ⟵
- complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

LTL model checking problem

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LTL-formula φ over *AP*

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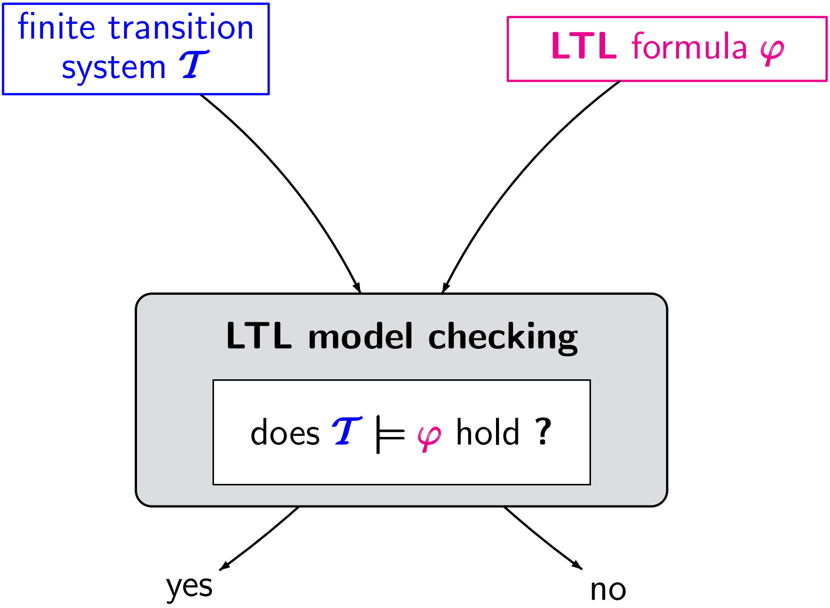
↑

construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets the acceptance condition of $\mathcal{A}$

Automata-based LTL model checking



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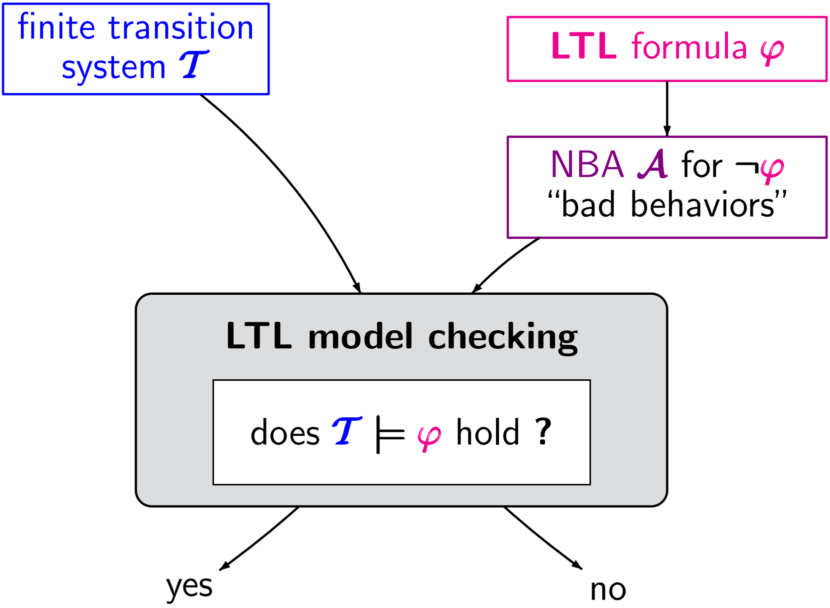

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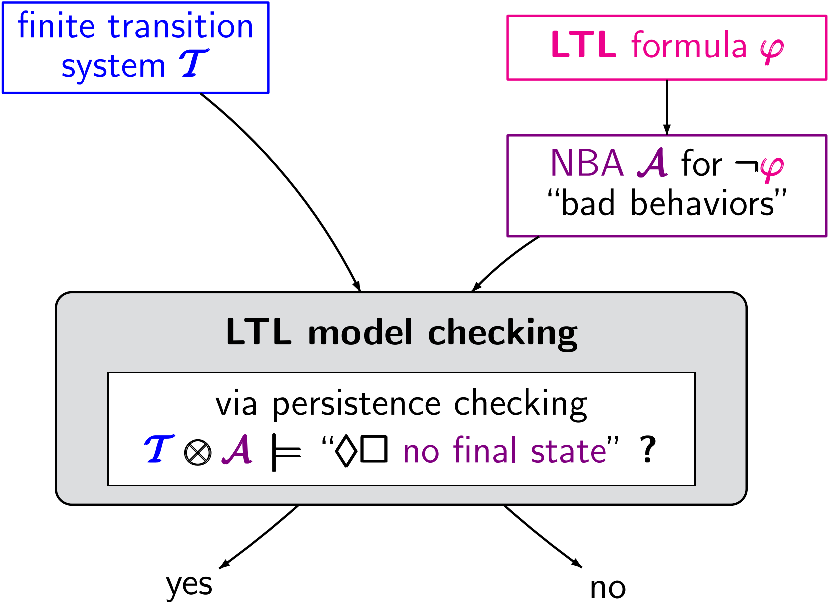

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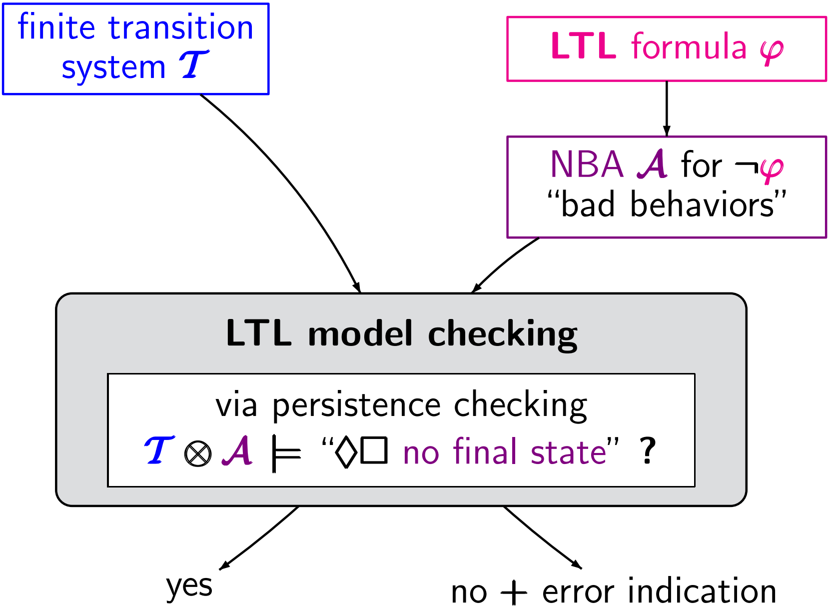

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iff $\mathcal{T} \otimes \mathcal{A} \models \Diamond \Box \neg F$ ⟵ persistence checking

From LTL to NBA

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For each **LTL** formula φ over AP there is an **NBA** $\mathcal{A}$ over the alphabet 2^{AP} such that

- $Words(\varphi) = \mathcal{L}_\omega(\mathcal{A})$
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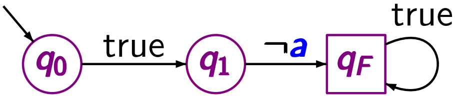
proof: ... later ...



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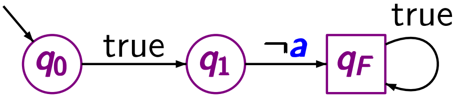
NBA for LTL formulas



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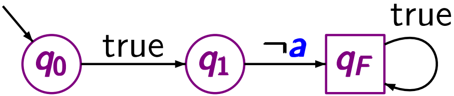

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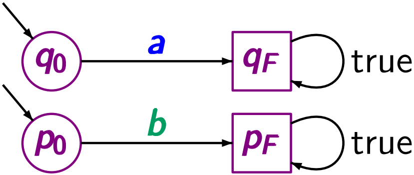

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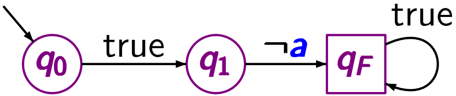

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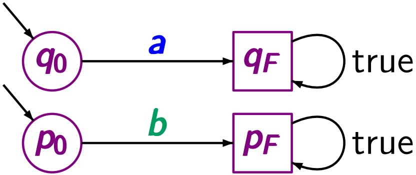

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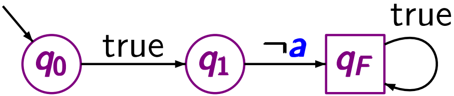

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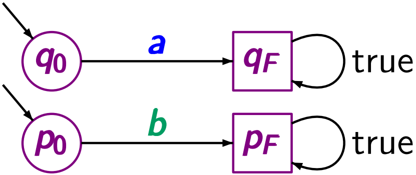

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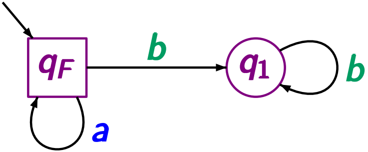

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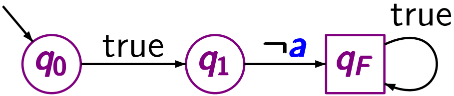

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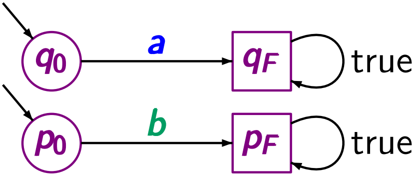

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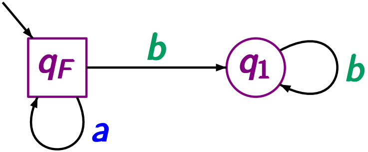

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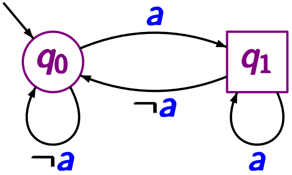

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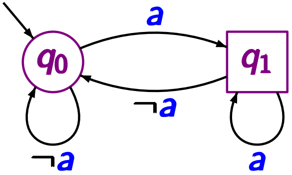

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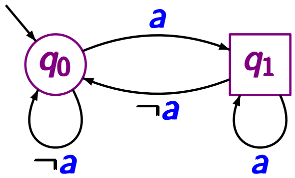

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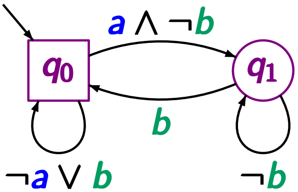

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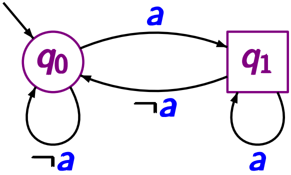

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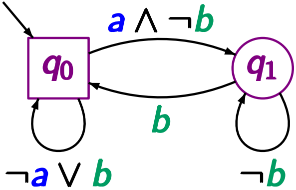

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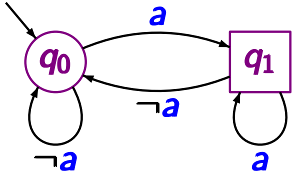

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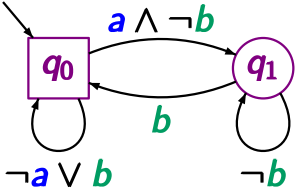

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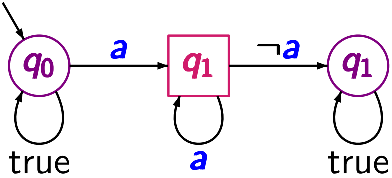

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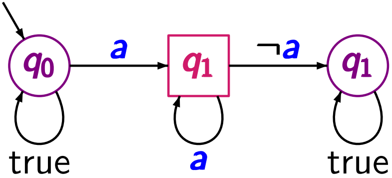

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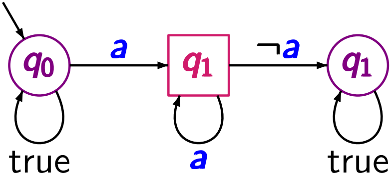

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possible runs for $\{a\}^\omega$

$q_0\ q_0\ q_0\ q_0\ q_0\ q_0\ \ldots$	not accepting
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$\vdots$	

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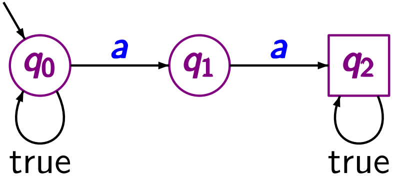

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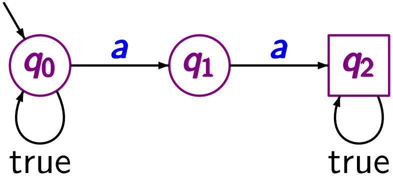

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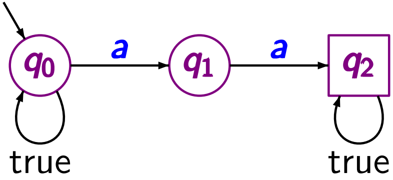

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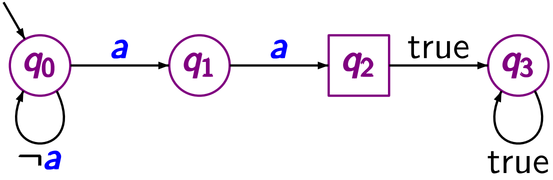

$\mathcal{L}_\omega(\mathcal{A}) = \varnothing$

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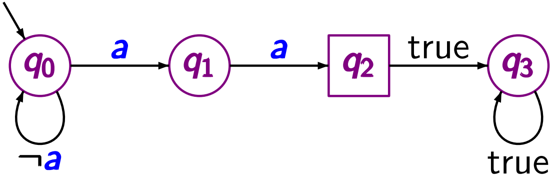
even if $\mathcal{A}$ is a non-blocking DFA

Example: $E \mathrel{\widehat{=}}$ "never a twice in a row"



$\mathcal{L}_\omega(\mathcal{A}) = \varnothing$

NFA and NBA for safety properties

Let $\mathcal{A}$ be an **NFA** for the language of all bad prefixes for a safety property E. Then:

$$\mathcal{L}_\omega(\mathcal{A}) = \overline{E} = (2^{AP})^\omega \setminus E = Words(\neg\varphi)$$

wrong, if $\mathcal{L}(\mathcal{A})$ = language of minimal bad prefixes even if $\mathcal{A}$ is a non-blocking DFA

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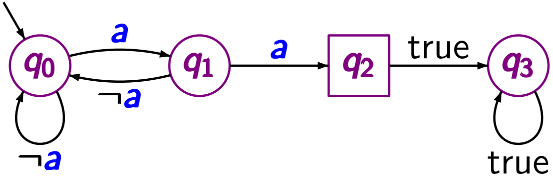

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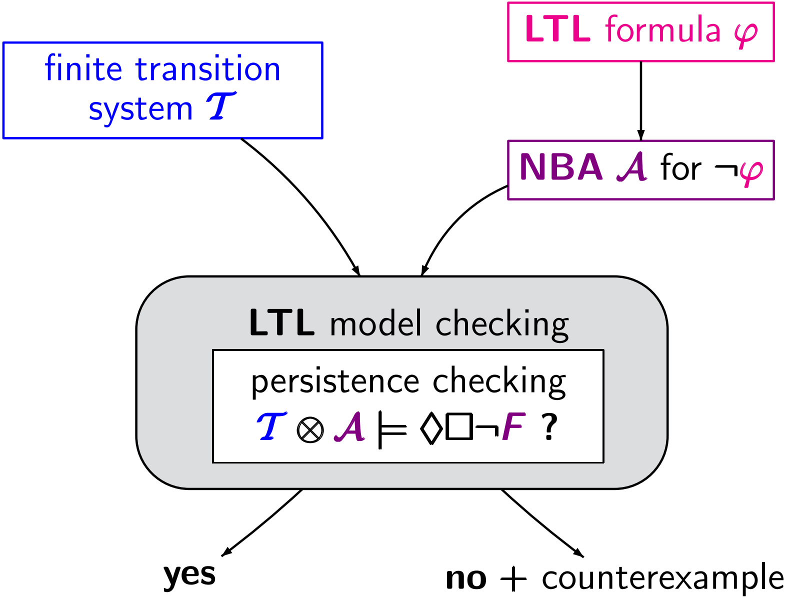
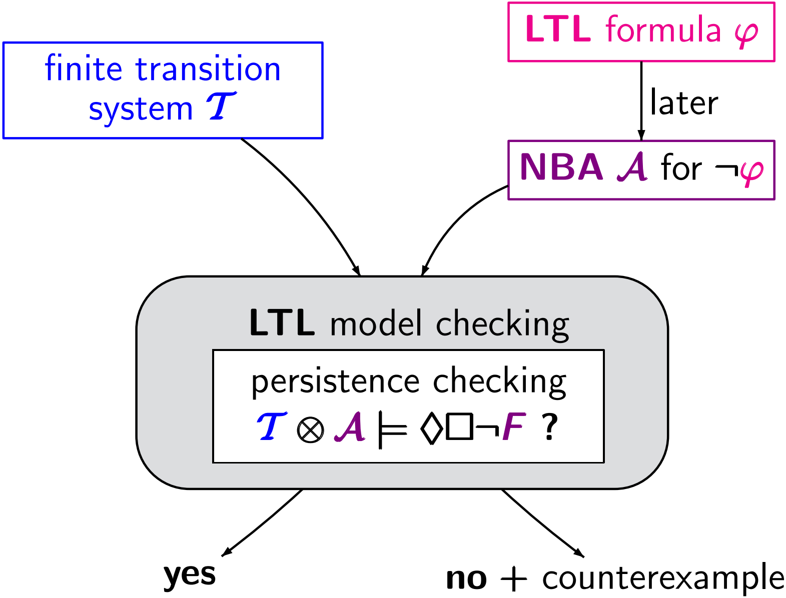


finite transition system $\mathcal{T}$
LTL formula φ
NBA $\mathcal{A}$ for $\neg\varphi$
LTL model checking
persistence checking
$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?
yes
no + counterexample



LTL formula φ
later
finite transition system $\mathcal{T}$
NBA $\mathcal{A}$ for $\neg\varphi$
LTL model checking
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$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?
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LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

LTL model checking

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construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ $\longleftarrow$ persistence checking nested **DFS** *given:* finite TS $\mathcal{T}$, LTL-formula φ

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IF $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

THEN return "**yes**"

ELSE compute a counterexample

$\langle s_0, p_0 \rangle \ldots \langle s_n, p_n \rangle \ldots \langle s_n, p_n \rangle$

for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$

return "**no**" and $s_0 \ldots s_n \ldots s_n$

Complexity of LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold ?

~~construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$~~

check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ ⟵ persistence checking nested **DFS**

IF $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

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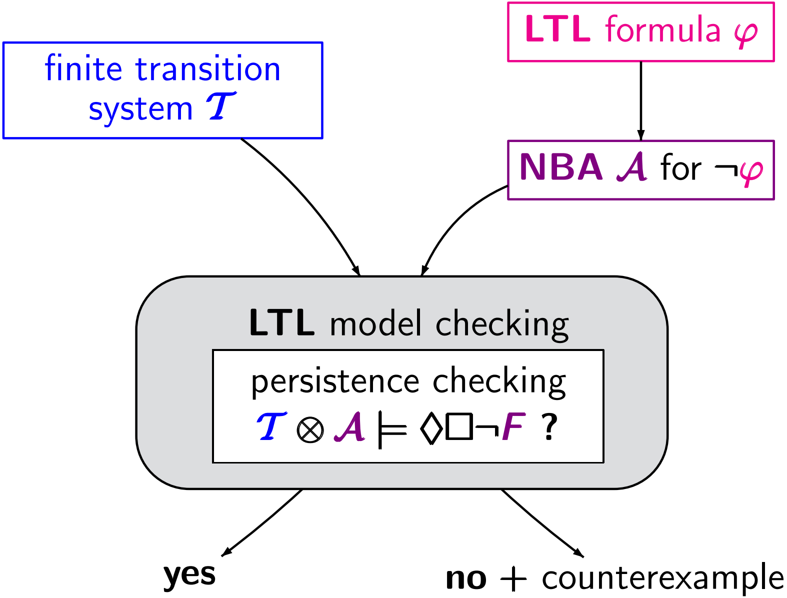
for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$

return "**no**" and $s_0 \ldots s_n \ldots s_n$

time complexity: $\mathcal{O}(size(\mathcal{T}) \cdot size(\mathcal{A}))$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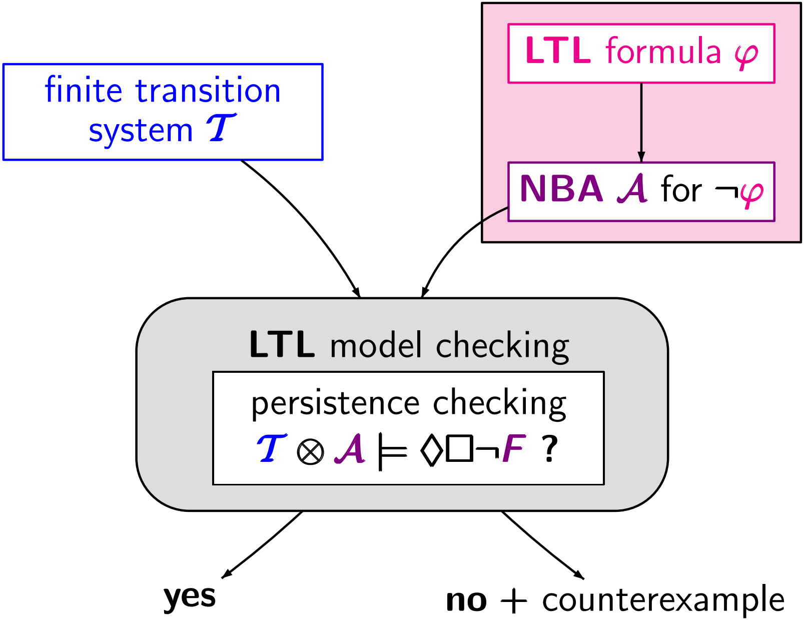

From LTL to NBA

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For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$$

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LTL formula φ

↓

NBA $\mathcal{A}$ s.t.
$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$

nondeterministic
Büchi automaton

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LTL formula φ

↓

GNBA $\mathcal{G}$ s.t. $\mathcal{L}_\omega(\mathcal{G}) = \mathit{Words}(\varphi)$ — generalized NBA, several acceptance sets

↓

NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$ — nondeterministic Büchi automaton, **1** acceptance set

For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$$

LTL formula φ

↓

GNBA $\mathcal{G}$ s.t.
$\mathcal{L}_\omega(\mathcal{G}) = \mathit{Words}(\varphi)$

generalized NBA
$\boldsymbol{k}$ acceptance sets

↓

$\boldsymbol{k}$ copies of $\mathcal{G}$

NBA $\mathcal{A}$ s.t.
$\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

nondeterministic
Büchi automaton
1 acceptance set

Encoding of LTL semantics in a GNBA

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idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

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semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	
next $\bigcirc$	
until **U**	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until **U**	via expansion law

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idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until **U**	via expansion law

$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

encoded in the states (ψ_2, ψ_1)

encoded in the transition relation ($\bigcirc$)

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until U	expansion law, **least fixed point**

$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

encoded in the states

encoded in the transition relation

acceptance condition

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for *Words*(φ)

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$$A_0 \quad A_1 \quad A_2 \quad A_3 \quad \ldots \quad \in Words(\varphi)$$

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$$
\begin{array}{cccccl}
A_0 & A_1 & A_2 & A_3 & \ldots & \in Words(\varphi) \\
\downarrow & \downarrow & \downarrow & \downarrow & & \\
B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run}
\end{array}
$$

where $B_i = \{\psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\}$

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$$\begin{array}{ccccccl} A_0 & A_1 & A_2 & A_3 & \ldots & \in Words(\varphi) \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run} \end{array}$$

where $B_i = \{\psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\}$

↑ set of subformulas of φ and their negations

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

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Example: $\varphi = a\, \mathsf{U}(\neg a \wedge b)$

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$$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

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$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \quad \models \varphi$$

$$\downarrow$$

$$\begin{pmatrix} a \\ \neg b \\ \neg\psi \\ \varphi \end{pmatrix}$$

just for better readability:
tuple rather than set notation

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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$$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$$\downarrow \qquad \downarrow$$

a	a
$\neg b$	$\neg b$
$\neg \psi$	$\neg \psi$
φ	φ

LTL $\rightsquigarrow$ GNBA

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$\downarrow \quad \downarrow \quad \downarrow$

a	a	a
$\neg b$	$\neg b$	b
$\neg\psi$	$\neg\psi$	$\neg\psi$
φ	φ	φ

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$				
a	a	a	$\neg a$				
$\neg b$	$\neg b$	b	b				
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ				
φ	φ	φ	φ				

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
a	a	a	$\neg a$	$\neg a$			
$\neg b$	$\neg b$	b	b	$\neg b$			
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ	$\neg \psi$			
φ	φ	φ	φ	$\neg \varphi$			

LTL $\rightsquigarrow$ GNBA

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Example: $\varphi = a\, \mathsf{U} (\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
a	a	a	$\neg a$	$\neg a$	$\neg a$		
$\neg b$	$\neg b$	b	b	$\neg b$	$\neg b$		
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ	$\neg \psi$	$\neg \psi$	$\ldots$	
φ	φ	φ	φ	$\neg \varphi$	$\neg \varphi$		

Closure of LTL formulas

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Let φ be an LTL formula. Then:

$$\mathit{subf}(\varphi) \stackrel{\text{def}}{=} \text{set of all subformulas of } \varphi$$

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where ψ and $\neg\neg\psi$ are identified

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Example: if $\varphi = a\,\mathsf{U}(\neg a \wedge b)$ then

$$cl(\varphi) = \{a, b, \neg a \wedge b, \varphi\} \cup \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$$

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Example: if $\varphi = a \, \mathsf{U} (\neg a \wedge b)$ then

$$cl(\varphi) = \{a, b, \neg a \wedge b, \varphi\} \cup \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$$

Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(true \, \mathsf{U} \neg a)$ then

$$cl(\varphi') = \{a, \neg a, true, \neg true, \Box a, \neg\Box a\}$$

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \mathit{true} \in cl(\varphi) & \text{implies} & \mathit{true} \in B \end{array}$$

(ii) B is locally consistent with respect to until U,
i.e., if $\psi_1 \mathsf{U} \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{l} \text{if } \psi_1 \mathsf{U} \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \mathsf{U} \psi_2 \in B \end{array}$$

Elementary or not?

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$

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$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary
propositional inconsistent

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$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary

propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$

Let $\varphi = a \,\mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ — not elementary, propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ — not elementary, not maximal

as $\neg a \wedge b \notin B_2$

$\neg(\neg a \wedge b) \notin B_2$

Elementary or not?

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ not elementary, not maximal
as $\neg a \wedge b \notin B_2$
$\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$

Elementary or not?

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$.

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propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ — not elementary, not maximal
as $\neg a \wedge b \notin B_2$
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$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ — not elementary
not locally consistent for U

Elementary or not?

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$.

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$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ not elementary
not locally consistent for U

$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Let $\varphi = a \, \mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$ not elementary
propositional inconsistent

$B_2 = \{\neg a, b, \varphi\}$ not elementary, not maximal
as $\neg a \wedge b \notin B_2$
$\neg(\neg a \wedge b) \notin B_2$

$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ not elementary
not locally consistent for **U**

$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$ elementary closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. $\mathbf{U}$

For $\varphi = a \,\mathbf{U}(\neg a \wedge b)$, the elementary sets are:

$\{\ a,\ \ b, \neg(\neg a \wedge b), \varphi\}$ $\quad$ $\{\ a,\ \ b, \neg(\neg a \wedge b), \neg\varphi\}$

$\{\ a, \neg b, \neg(\neg a \wedge b), \varphi\}$ $\quad$ $\{\ a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

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Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$:

semantics of ...	encoding
propositional logic *true*, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until U	expansion law, least fixed point

$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

- ψ_2, ψ_1: encoded in the states
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if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

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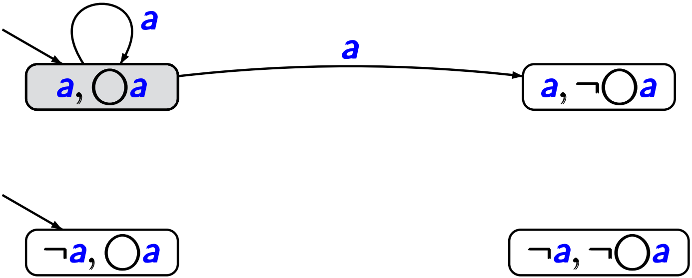

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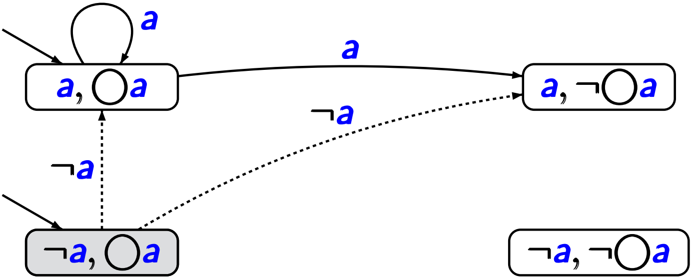

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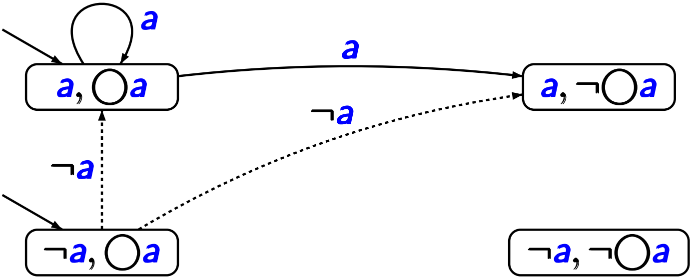

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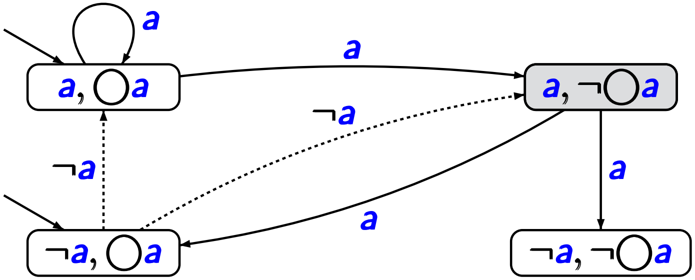

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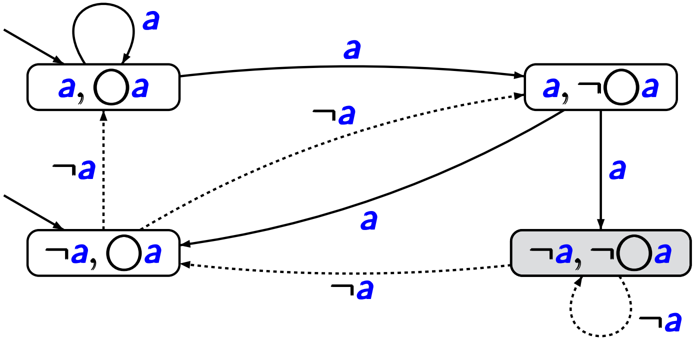

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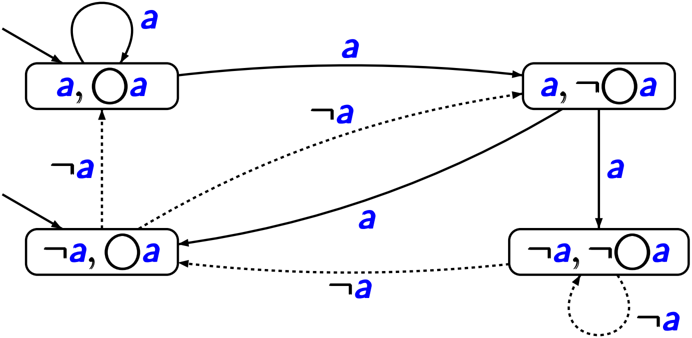
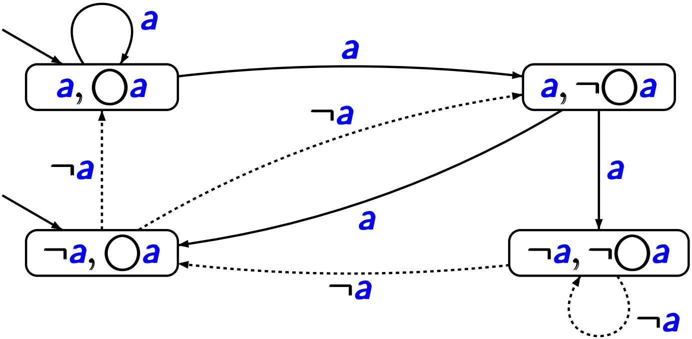


set of acceptance sets:

Example: GNBA for $\varphi = \bigcirc a$



set of acceptance sets: $\mathcal{F} = \varnothing$

hence: all words having an infinite run are accepted

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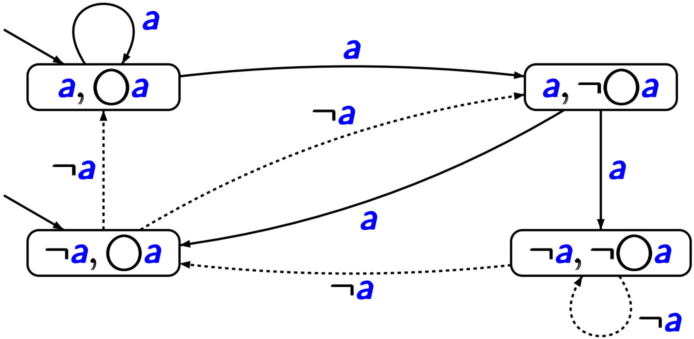

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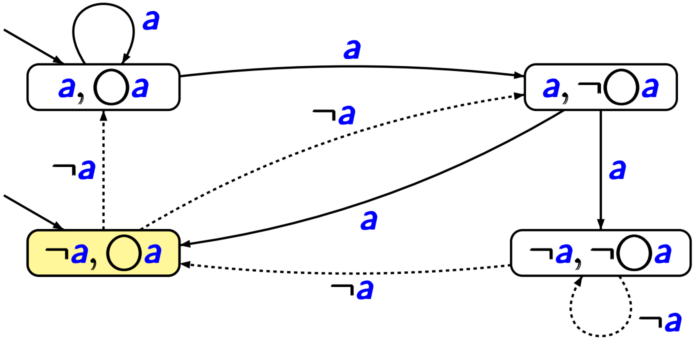

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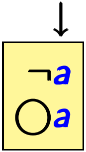

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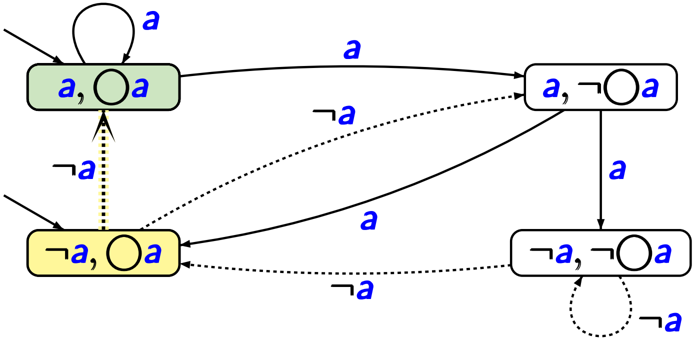

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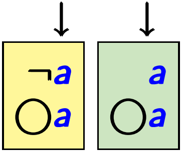

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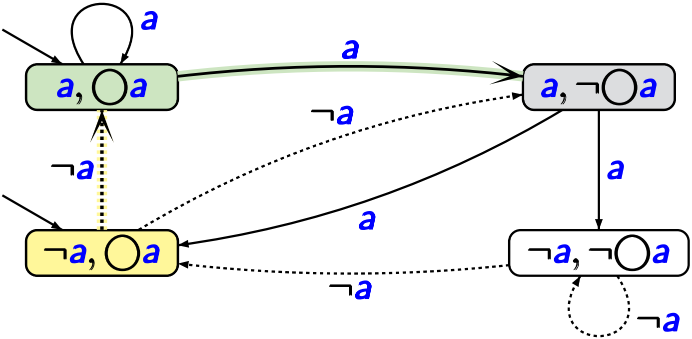

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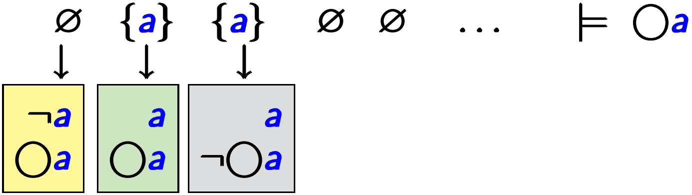

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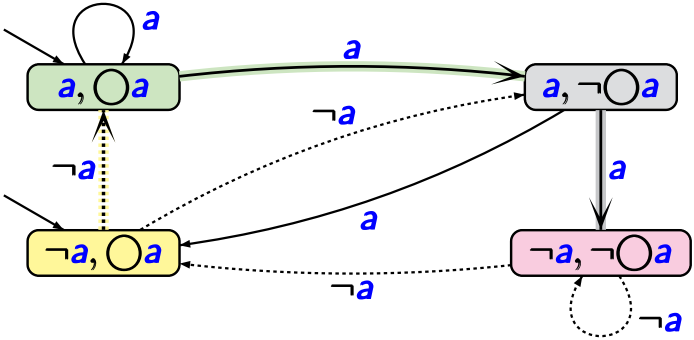

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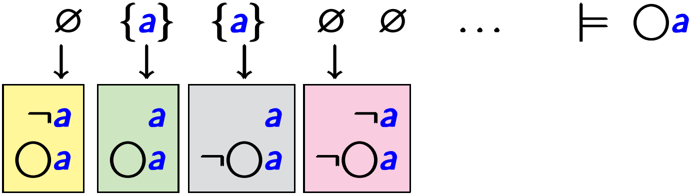

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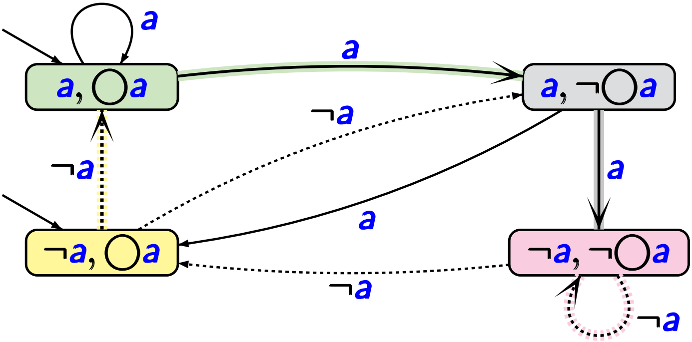

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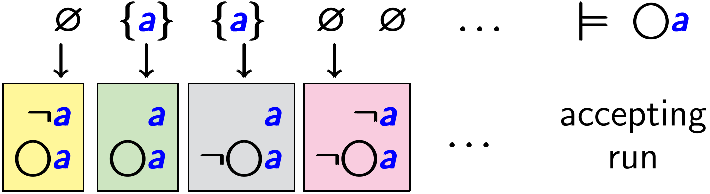
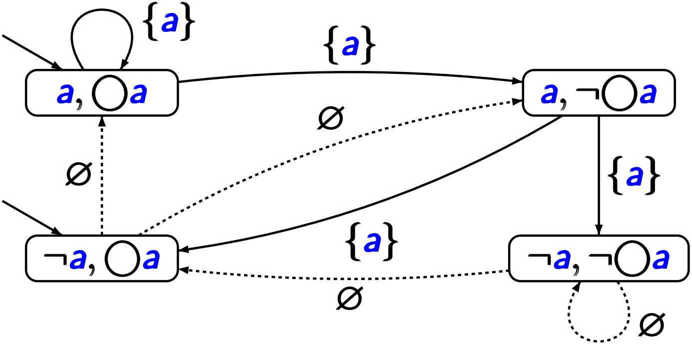


Soundness of the GNBA for $\varphi = \bigcirc a$



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

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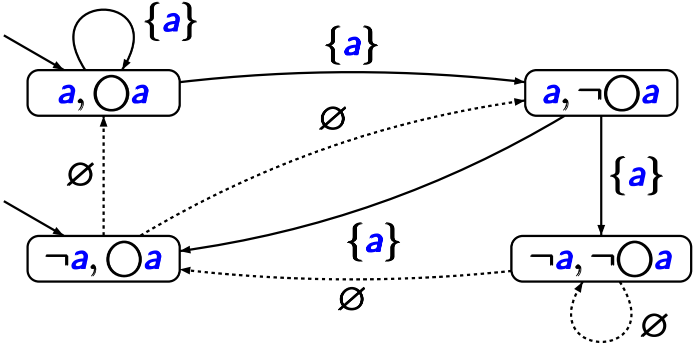

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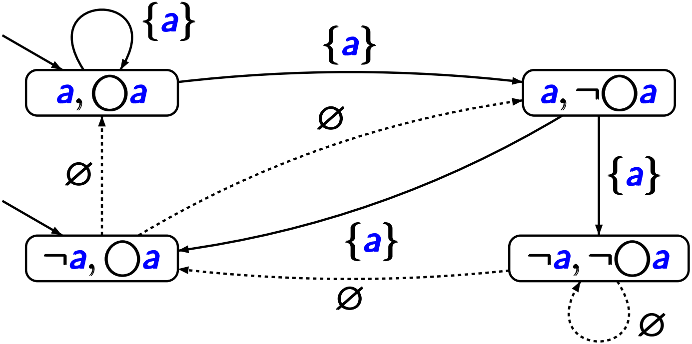

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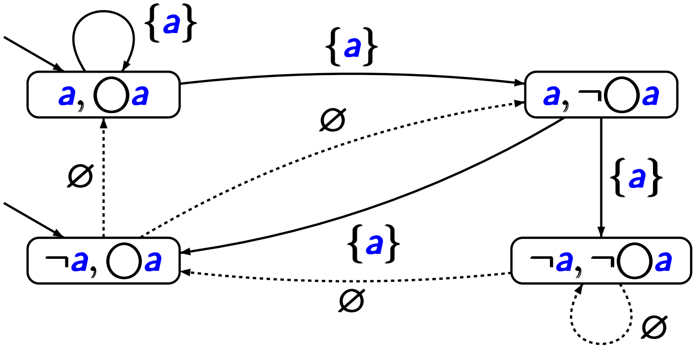

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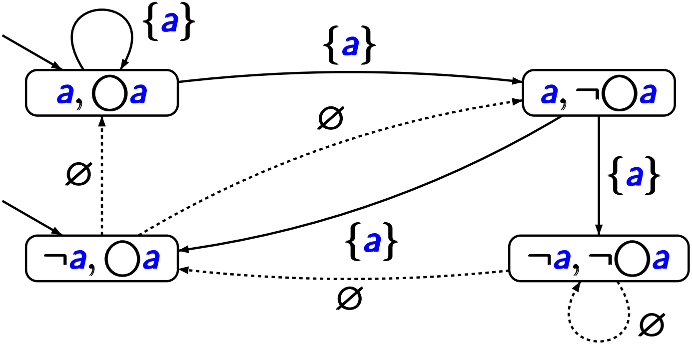

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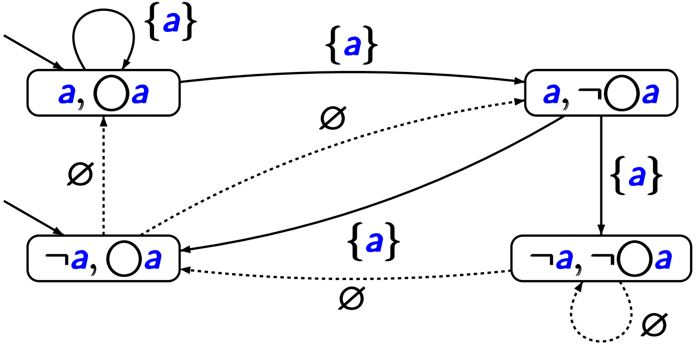

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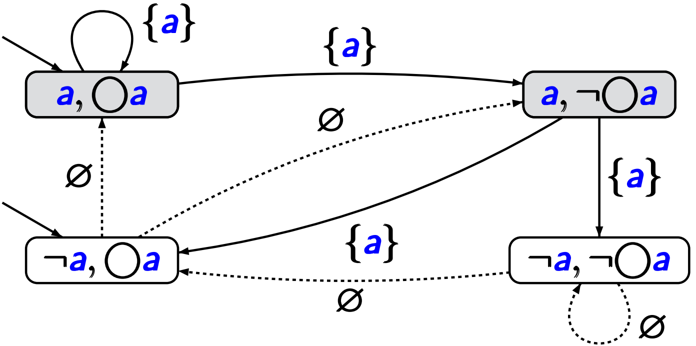

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$\Longrightarrow$ $\{a\} = B_1 \cap AP = A_1$

Example: GNBA for $\varphi = a \, \mathsf{U} \, b$

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$a, b, a \, \mathsf{U} \, b$

$\neg a, \neg b, \neg(a \, \mathsf{U} \, b)$

$a, \neg b, a \, \mathsf{U} \, b$

$a, \neg b, \neg(a \, \mathsf{U} \, b)$

$\neg a, b, a \, \mathsf{U} \, b$

locally inconsistent: $\{a, b, \neg(a \, \mathsf{U} \, b)\}$
$\{\neg a, b, \neg(a \, \mathsf{U} \, b)\}$
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$\neg a, b, a \,\mathsf{U}\, b$

initial states: B with $\varphi = a \,\mathsf{U}\, b \in B$

Example: GNBA for $\varphi = a \mathsf{U} b$

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acceptance condition: just one set of accept states

F = set of all B with $\varphi \notin B$ or $b \in B$

Example: GNBA for $\varphi = a \,\mathsf{U}\, b$ ⟵ NBA

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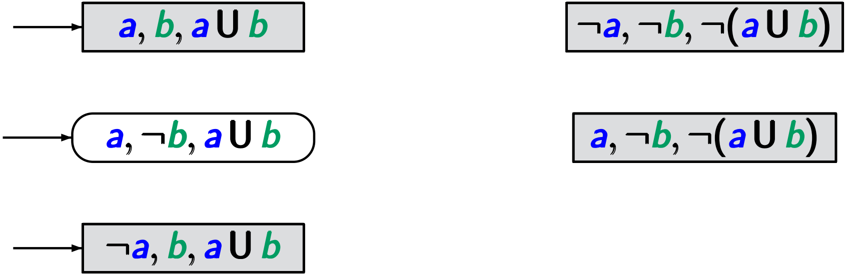

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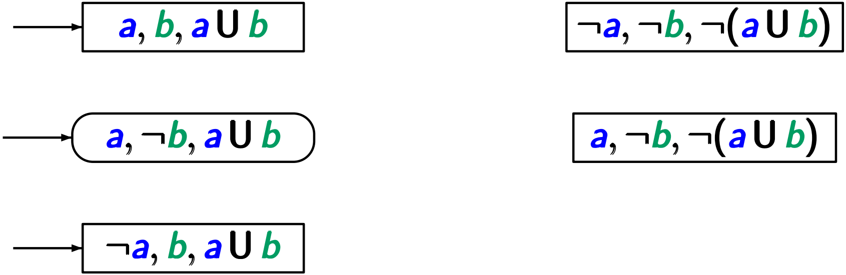
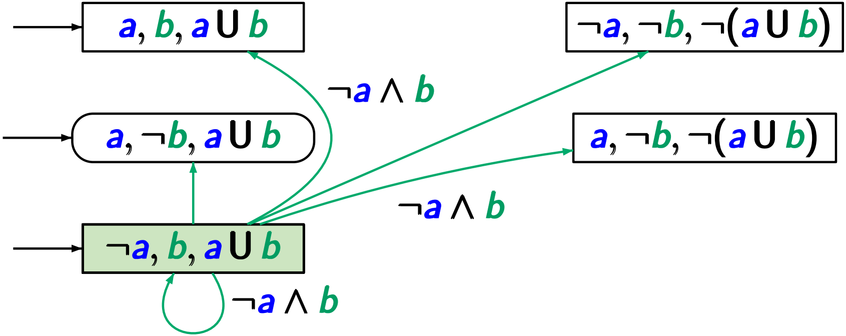


transition relation: $B' \in \delta(B, B \cap AP)$ iff

$$a \,\mathsf{U}\, b \in B \iff \big(b \in B \;\vee\; (\, a \in B \;\wedge\; a \,\mathsf{U}\, b \in B'\,)\big)$$

Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$



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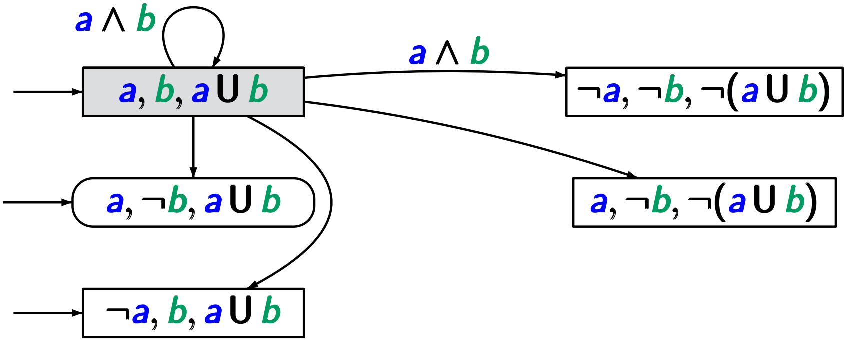

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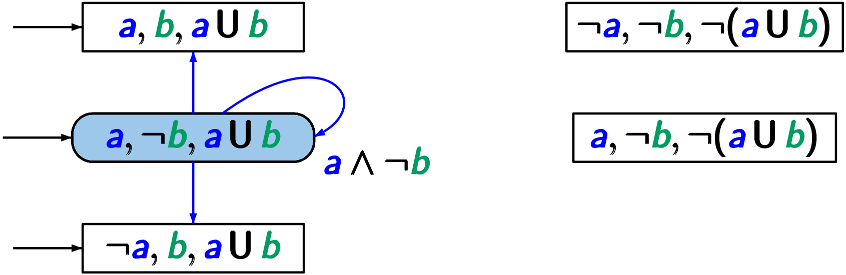

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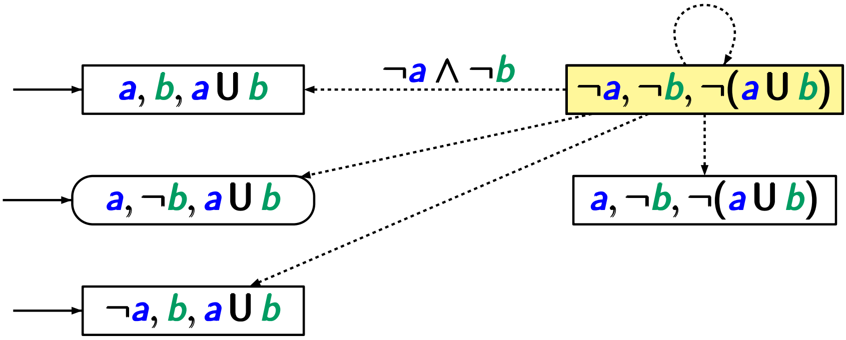

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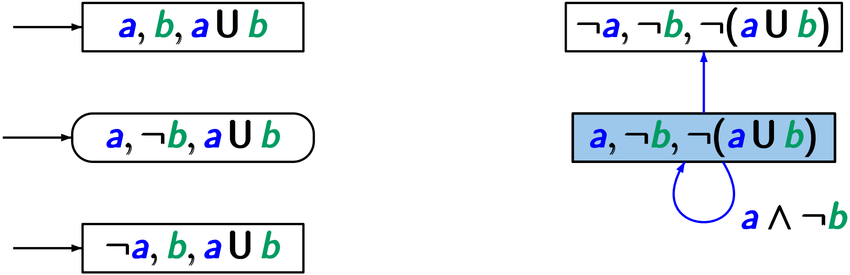
$$a \,\mathsf{U}\, b \in B \iff \big(b \in B \vee (a \in B \wedge a \,\mathsf{U}\, b \in B')\big)$$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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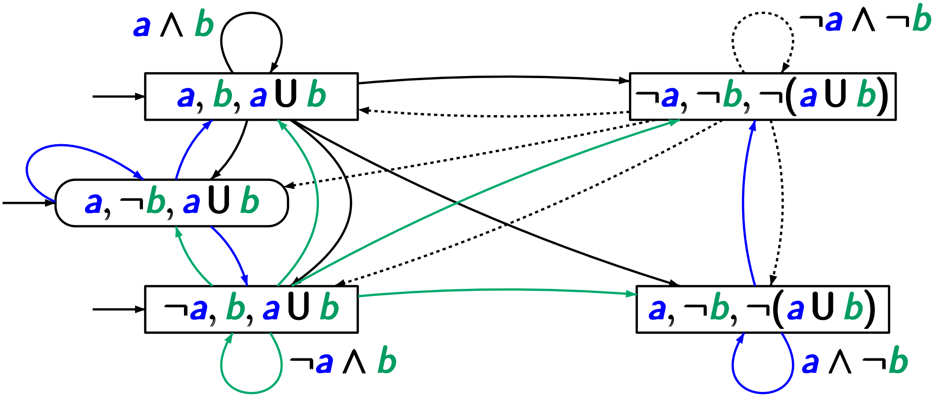

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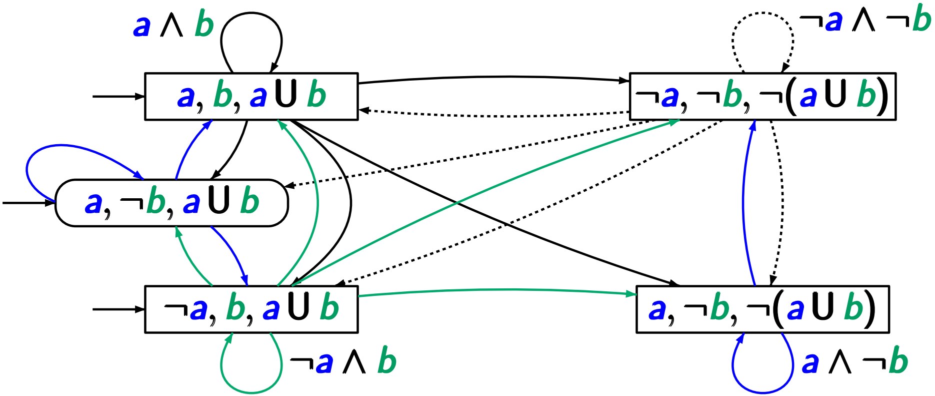

$\{a\} \quad \{a\} \quad \{a,b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \, \mathsf{U} \, b$

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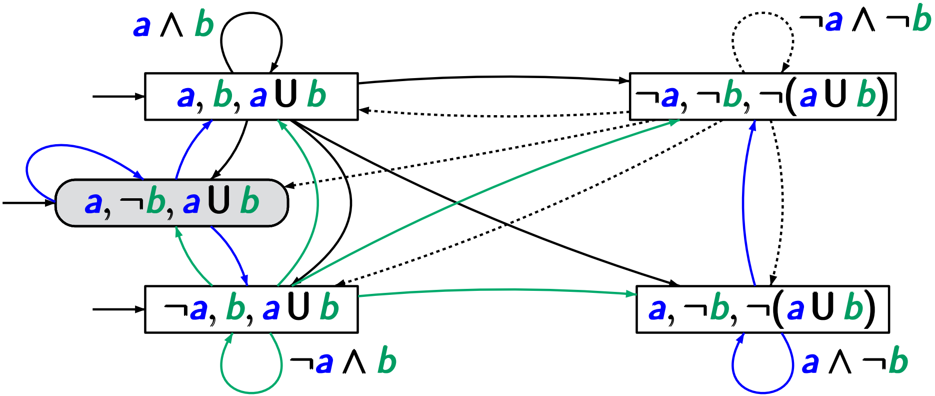

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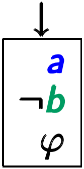

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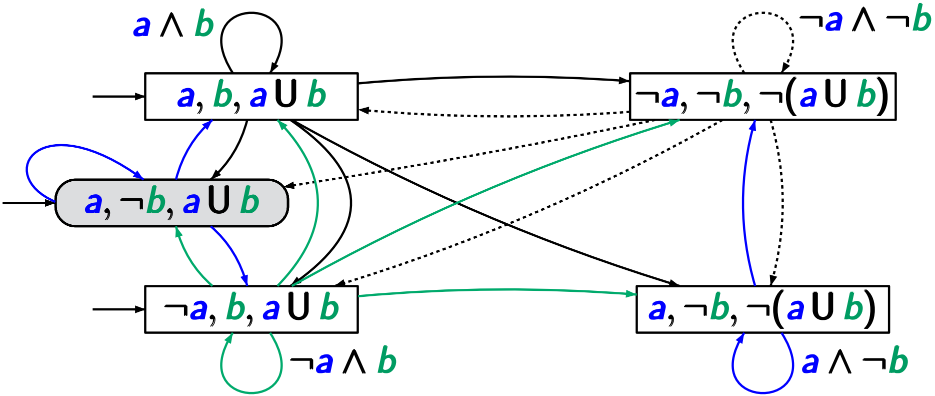

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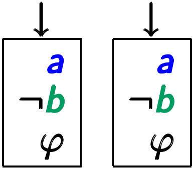

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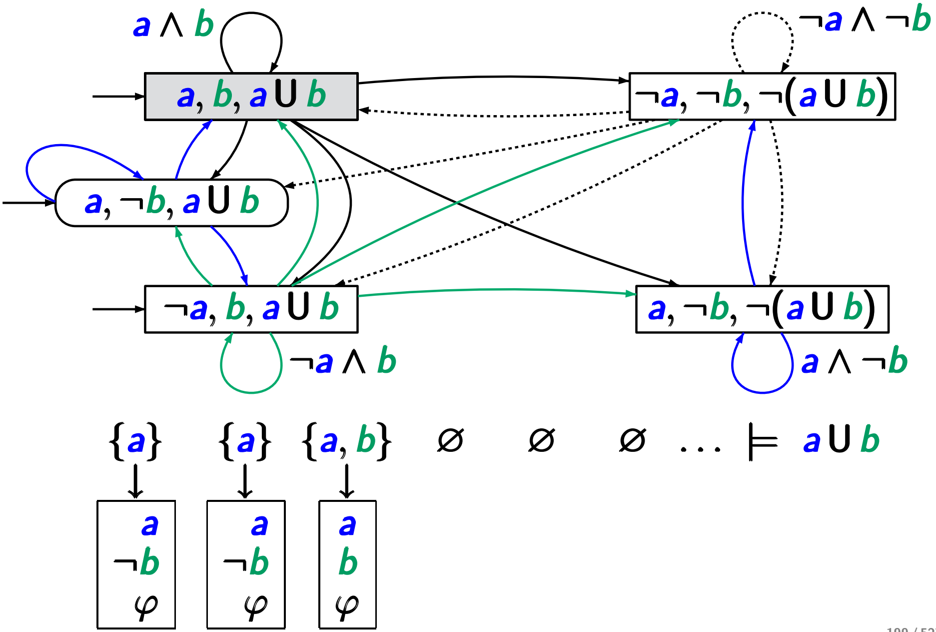

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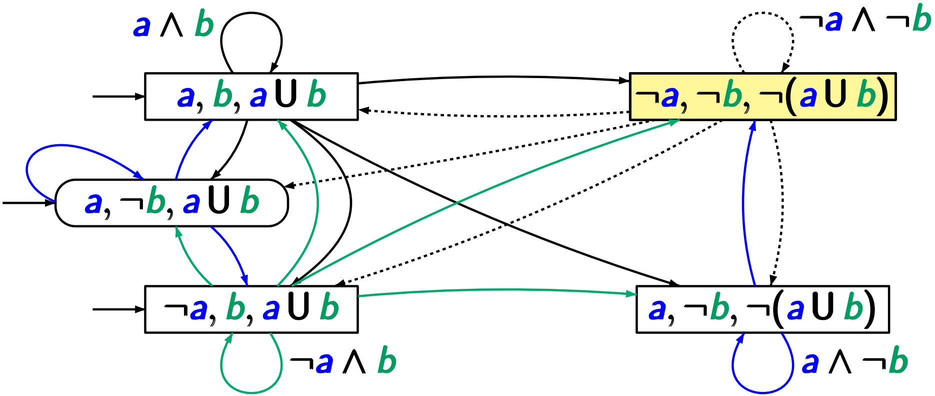

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$\{a\}$	$\{a\}$	$\{a,b\}$	$\varnothing$
a	a	a	$\neg a$
$\neg b$	$\neg b$	b	$\neg b$
φ	φ	φ	$\neg\varphi$

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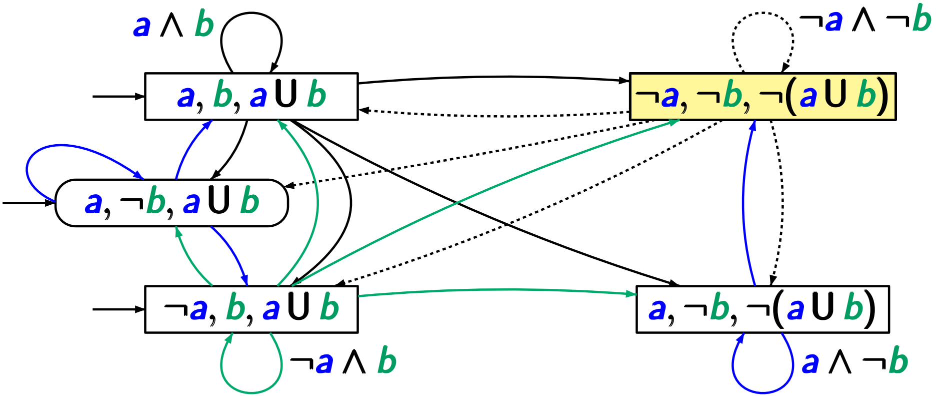
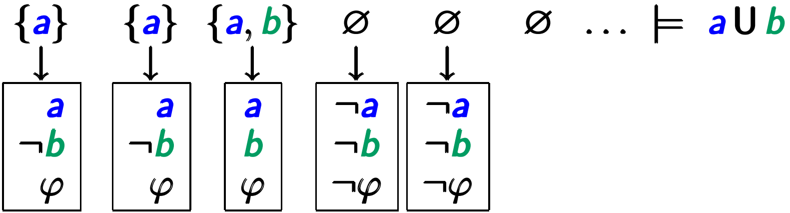
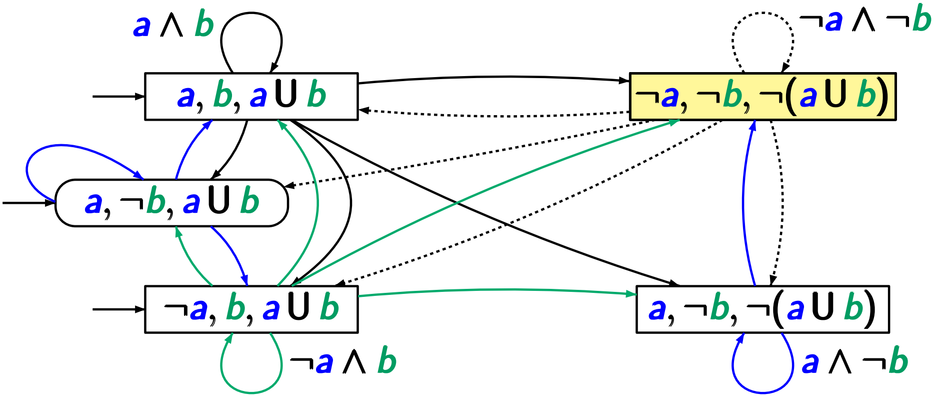
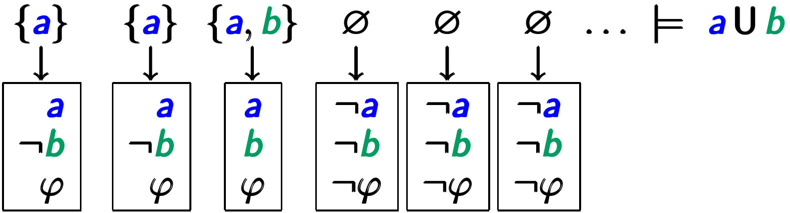




Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$





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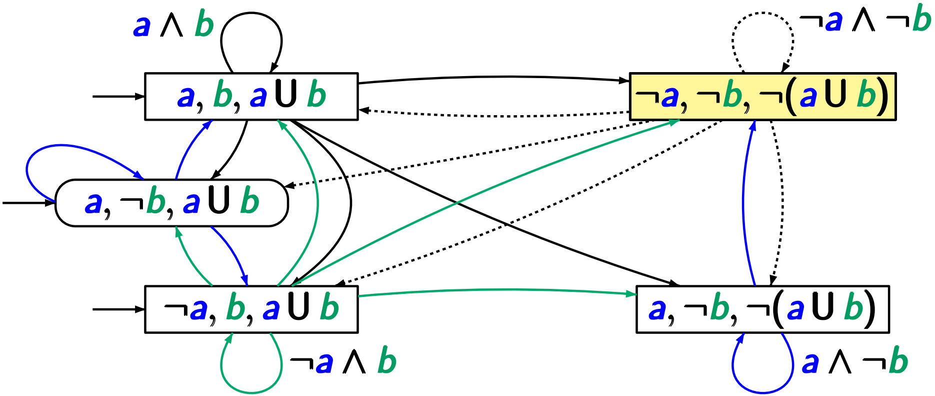
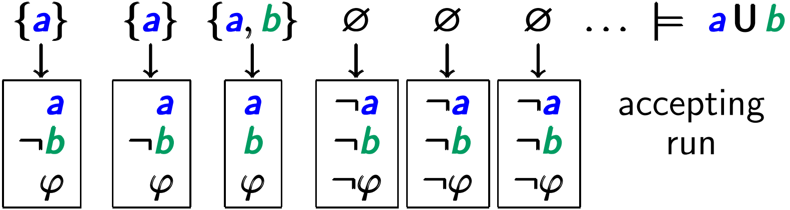
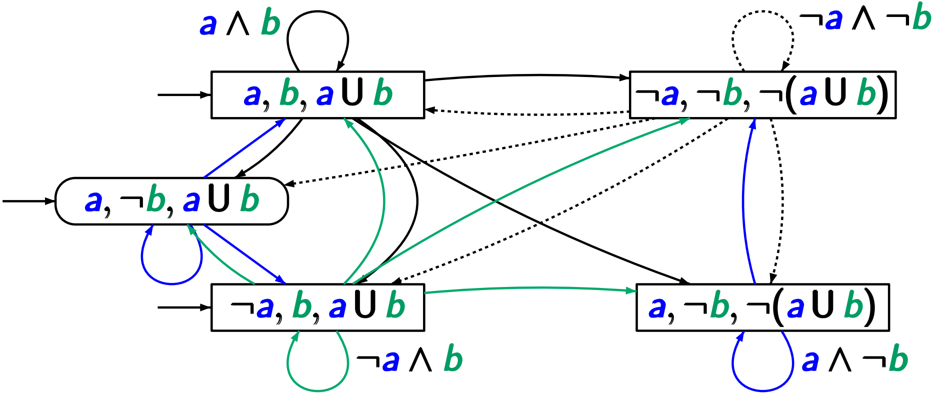




accepting run

Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$



$\{a\}\,\{a\}\,\{a\}\,\{a\}\,\ldots \not\models \varphi$

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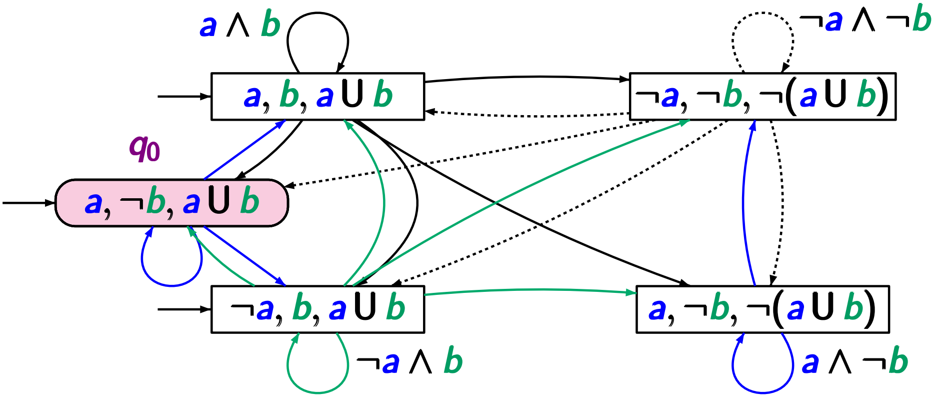

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only **1** infinite run: $q_0\, q_0\, q_0\, \ldots$

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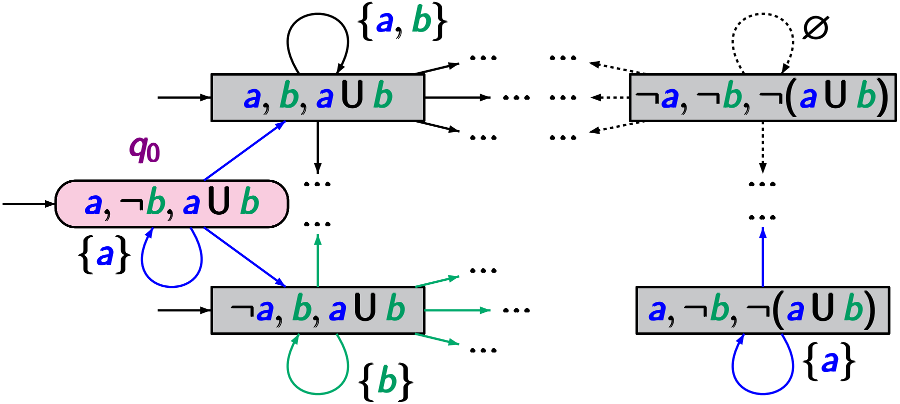

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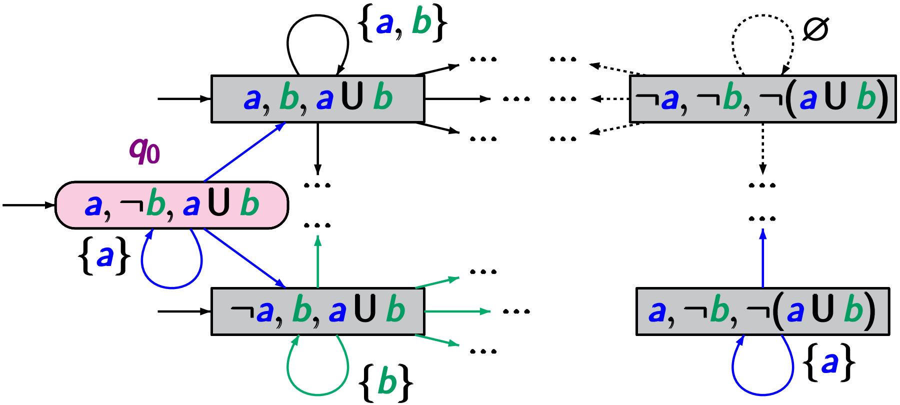

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only **1** infinite run: $q_0 \, q_0 \, q_0 \, \ldots$ not accepting

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Soundness

.... of the construction LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

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Claim: $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\, A_1\, A_2 \ldots \in (2^{AP})^\omega$

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Accepting runs for the elements of *Words*(φ)

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ elementary formula-sets $B \subseteq cl(\varphi)$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

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Example: $\varphi = a \mathsf{U} (\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are states in $\mathcal{G}$, i.e., elementary subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

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$$\downarrow$$

$$\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}}$$

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$\downarrow$ $\qquad \downarrow$

$\{a\}$	$\{a\}$
a	a
$\neg b$	$\neg b$
$\neg\psi$	$\neg\psi$
φ	φ

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$\downarrow \quad \downarrow \quad \downarrow$

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a	a	a
$\neg b$	$\neg b$	b
$\neg\psi$	$\neg\psi$	$\neg\psi$
φ	φ	φ

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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$				
a	a	a	$\neg a$				
$\neg b$	$\neg b$	b	b				
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ				
φ	φ	φ	φ				

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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
a	a	a	$\neg a$	$\neg a$			
$\neg b$	$\neg b$	b	b	$\neg b$			
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φ	φ	φ	φ	$\neg \varphi$			

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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
a	a	a	$\neg a$	$\neg a$	$\neg a$		
$\neg b$	$\neg b$	b	b	$\neg b$	$\neg b$		
$\neg \psi$	$\neg \psi$	$\neg \psi$	ψ	$\neg \psi$	$\neg \psi$	$\ldots$	
φ	φ	φ	φ	$\neg \varphi$	$\neg \varphi$		

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Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$\psi \notin B$	iff	$\neg\psi \in B$
$\psi_1 \wedge \psi_2 \in B$	iff	$\psi_1 \in B$ and $\psi_2 \in B$
$\mathit{true} \in cl(\varphi)$	implies	$\mathit{true} \in B$

(ii) B is locally consistent with respect to until $\mathbf{U}$,
i.e., if $\psi_1 \,\mathbf{U}\, \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \,\mathbf{U}\, \psi_2 \in B$

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0 A_1 A_2 \ldots \in (2^{AP})^\omega$

with $A_0 A_1 A_2 \ldots \models \varphi$

has an accepting run in $\mathcal{G}$

"$\supseteq$" show: for all infinite words $A_0 A_1 A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq \mathit{Words}(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

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The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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↑ as $B_0 \in Q_0$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq \mathit{Words}(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \stackrel{\infty}{\exists} j \geq 0.\ B_j \in F \qquad (*)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

$\Longrightarrow$ there is an accepting run $B_0\, B_1\, B_2 \ldots$ for σ

$\Longrightarrow$ $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t. $\varphi \in B_0$ and $(*)$ holds

as $B_0 \in Q_0$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F \qquad (*)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

$\Longrightarrow$ there is an accepting run $B_0\, B_1\, B_2 \ldots$ for σ

$\Longrightarrow$ $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t. $\varphi \in B_0$ (as $B_0 \in Q_0$)
and $(*)$ holds

$\Longrightarrow$ $\sigma = A_0\, A_1\, A_2 \ldots \models \varphi$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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The claim yields that for each $\sigma = A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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$\Longrightarrow$ $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t. $\varphi \in B_0$
and $(*)$ holds

$\uparrow$ as $B_0 \in Q_0$

$\Longrightarrow$ $\sigma = A_0\, A_1\, A_2 \ldots \models \varphi$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

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Proof by structural induction on ψ

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Proof by structural induction on ψ

base of induction:

$\psi = true$

$\psi = a \in AP$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq \mathit{Words}(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F \qquad (*)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Proof by structural induction on ψ

base of induction:

$\psi = \mathit{true}$

$\psi = a \in AP$

induction step:

$\psi = \neg\psi'$

$\psi = \psi_1 \wedge \psi_2$

$\psi = \bigcirc\psi'$

$\psi = \psi_1 \,\mathsf{U}\, \psi_2$

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$.

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$

note: ***true*** is contained in all elementary formula-sets

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

note: ***true*** is contained in all elementary formula-sets
true holds for all paths/traces

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \; \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

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Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

$$A_0\, A_1\, A_2 \ldots \models \textit{true}$$

Let $\psi = a \in AP$.

Base of induction

Claim: If $B_0 \stackrel{A_0}{\rightarrow} B_1 \stackrel{A_1}{\rightarrow} B_2 \stackrel{A_2}{\rightarrow} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \stackrel{\infty}{\exists} j \geq 0.\, B_j \in F$$

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Let $\psi = a \in AP$. Then:

$a \in B_0$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

Let $\psi = a \in AP$. Then:

$$a \in B_0 \iff a \in A_0$$

Base of induction

Claim: If $\boxed{B_0 \xrightarrow{A_0} B_1} \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F}\ \overset{\infty}{\exists}\, j \geq 0.\ B_j \in F \qquad A_0 = B_0 \cap AP$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

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Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

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Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\,A_1\,A_2 \ldots \models \textit{true}$

Let $\psi = a \in AP$. Then:

$$a \in B_0 \iff a \in A_0 \iff A_0\,A_1\,A_2 \ldots \models a$$

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} \, j \geq 0. \, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$\psi \in B_0$

iff $\psi' \notin B_0$ (maximal consistency)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \neg\psi'$:

$\psi \in B_0$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

iff $A_0\, A_1\, A_2 \ldots \models \psi$ (semantics of $\neg$)

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\psi \notin B \quad \text{iff} \quad \neg\psi \in B$$

$$\psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B$$

$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until U, i.e., if $\psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)$ then:

$$\text{if } \psi_1 \,\mathsf{U}\, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B$$

$$\text{if } \psi_2 \in B \text{ then } \psi_1 \,\mathsf{U}\, \psi_2 \in B$$

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
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$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until $\mathbf{U}$,
i.e., if $\psi_1 \, \mathbf{U} \, \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \, \mathbf{U} \, \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \, \mathbf{U} \, \psi_2 \in B$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

iff $A_0\, A_1\, A_2 \ldots \models \psi$ (semantics of $\wedge$)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \; \overset{\infty}{\exists} j \geq 0. \, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 \, A_1 \, A_2 \, \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

GNBA for LTL-formula φ

$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$\bigcirc\psi \in B$ iff $\psi \in B'$

$\psi_1 \mathsf{U} \psi_2 \in B$ iff $(\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \stackrel{\infty}{\exists} j \geq 0.\, B_j \in F$$

$$B_1 \in \delta(B_0, A_0)$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc\psi'$:

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iff $\psi' \in B_1$ (definition of δ)

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Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{B_1 \in \delta(B_0, A_0)}$$

then for all formulas $\psi \in cl(\varphi)$:

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iff $A_1\, A_2\, A_3 \ldots \models \psi'$ (induction hypothesis)

iff $A_0\, A_1\, A_2\, A_3 \ldots \models \psi$ (semantics of $\bigcirc$)

Induction step: until

Recall: elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if $\psi, \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \textit{true} \in cl(\varphi) & \text{implies} & \textit{true} \in B \end{array}$$

(ii) B is locally consistent with respect to until $\mathbf{U}$, i.e., if $\psi_1 \mathbf{U} \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \mathbf{U} \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \mathbf{U} \psi_2 \in B$

Recall: GNBA for LTL-formula φ

$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$\bigcirc\psi \in B$ iff $\psi \in B'$

$\psi_1 \,\mathsf{U}\, \psi_2 \in B$ iff $(\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \,\mathsf{U}\, \psi_2 \in B')$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \,\mathsf{U}\, \psi_2 \notin B \vee \psi_2 \in B\}$

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$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

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Induction step for $\psi = \psi_1 \, \mathsf{U} \, \psi_2$:

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"$\Longleftarrow$": Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

$$A_j\, A_{j+1}\, A_{j+2} \ldots \models \psi_2$$
$$A_{j-1}\, A_j\, A_{j-1} \ldots \models \psi_1$$
$$A_{j-2}\, A_{j-1}\, A_j \ldots \models \psi_1$$
$$\vdots$$
$$A_0\, A_1\, A_2\, A_3 \ldots \models \psi_1$$

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A_{j-2}\, A_{j-1}\, A_j \ldots \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} \\
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B_j is elementary

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A_{j-1} A_j A_{j-1} \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge & \psi \in B_{j-1} \\
A_{j-2} A_{j-1} A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & & \\
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$$\begin{aligned} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow\ & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow\ & \psi \in B_2 \end{aligned}$$

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$$\left.\begin{array}{l} \quad \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow \psi \in B_2 \ \wedge \ \psi_2 \notin B_2 \\ \qquad \vdots \end{array}\right\} \Longrightarrow \forall j \geq 0.\ B_j \notin F_\psi \text{ where}$$

$$F_\psi = \{B : \psi \notin B \text{ or } \psi_2 \in B\}$$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \, \mathbf{U} \, \psi_2$:

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$$\left.\begin{array}{rl} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 \ \wedge \ \psi_2 \notin B_2 \\ & \vdots \end{array}\right\} \Longrightarrow \forall j \geq 0.\ B_j \notin F_\psi \text{ where}$$

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Contradiction!

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Induction step for $\psi = \psi_1 \, \mathsf{U} \, \psi_2$:

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$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$$\begin{array}{ll}
\neg\psi_2 & \in B_{j-1} \\
\neg\psi_2 & \in B_{j-2} \\
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\end{array}$$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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Induction step for $\psi = \psi_1 \mathsf{U} \psi_2$:

Let $\boxed{\psi \in B_0}$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \models \psi_2$$

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$$\Downarrow$$

$$A_0\, A_1\, A_2 \ldots \models \psi = \psi_1 \mathbf{U} \psi_2$$

Complexity: LTL $\rightsquigarrow$ NBA

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For each **LTL** formula φ, there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) \quad = \quad \mathit{Words}(\varphi)$$

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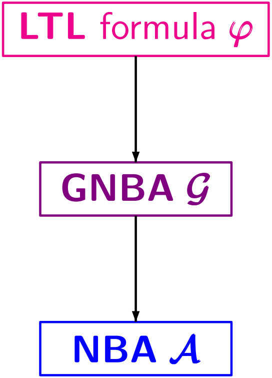

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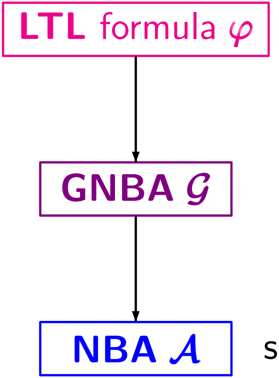

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LTL formula φ

↓

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GNBA $\mathcal{G}$ — size: $2^{|cl(\varphi)|}$

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Complexity: LTL $\rightsquigarrow$ NBA

For each **LTL** formula φ, there is an **NBA** $\mathcal{A}$ s.t.

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LTL formula φ

↓

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↓

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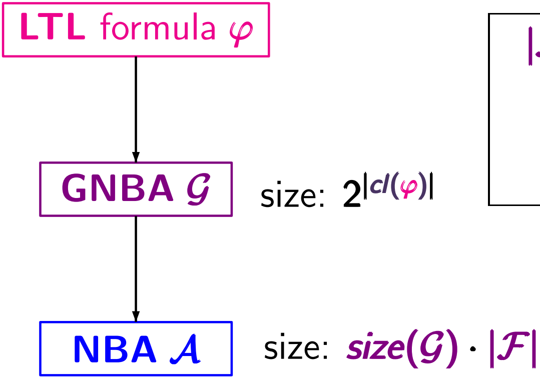

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Size of NBA for LTL formulas

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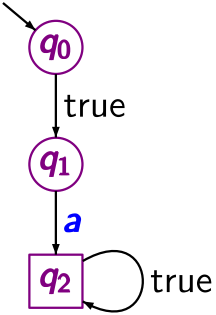
NBA for $\bigcirc a$



constructed GNBA has **4** states and **8** edges

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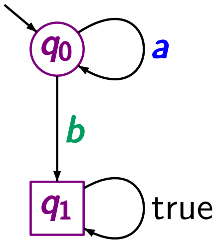
NBA for $a\,\mathbf{U}\,b$



constructed (G)NBA has **5** states and **20** edges

For the proposed transformation **LTL** $\rightsquigarrow$ **NBA**:

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... but there exists LTL formulas φ_n such that

- $|\varphi_n| = \mathcal{O}(poly(n))$
- each NBA for φ_n has at least 2^n states

LT-properties that have no "small" NBA

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consider the following family of LT-properties $(E_n)_{n \geq 1}$:

$$E_n = \left\{ \begin{array}{l} \text{set of all infinite words over } 2^{AP} \text{ of the form} \\ A_1\, A_2\, A_3 \ldots A_n\, A_1\, A_2\, A_3 \ldots A_n\, B_1\, B_2\, B_3\, B_4 \ldots \end{array} \right.$$

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LTL formula φ_n with $Words(\varphi_n) = E_n$

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LT-property E_n for $n=1$

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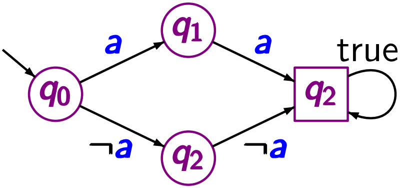
NBA for E_1 if $AP = \{a\}$:



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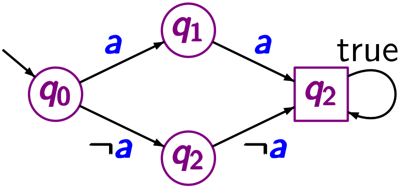

LTL-formula:
$a \leftrightarrow \bigcirc a$

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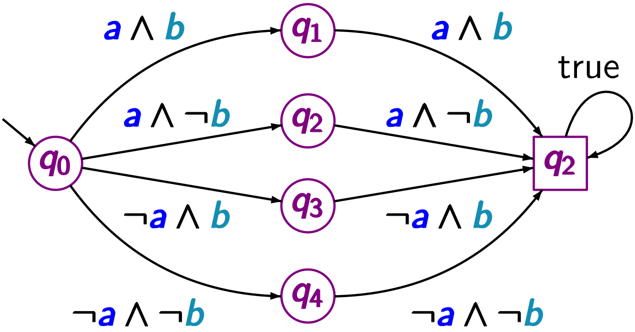

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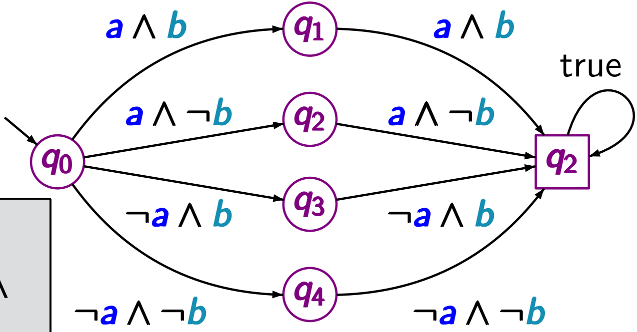
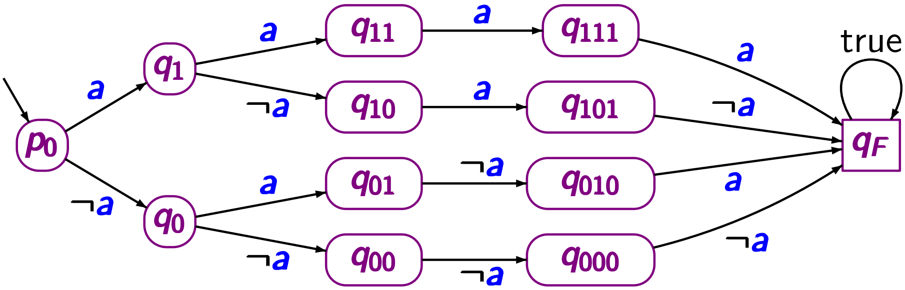


LTL-formula:

$(a \leftrightarrow \bigcirc a) \wedge$
$(b \leftrightarrow \bigcirc b)$

LT property E_n for n=2 and $AP = \{a\}$



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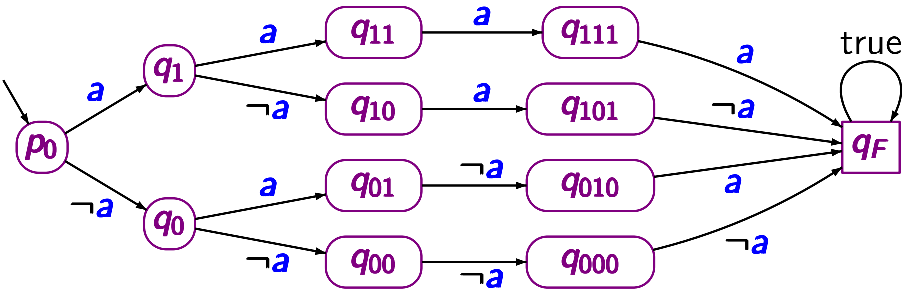

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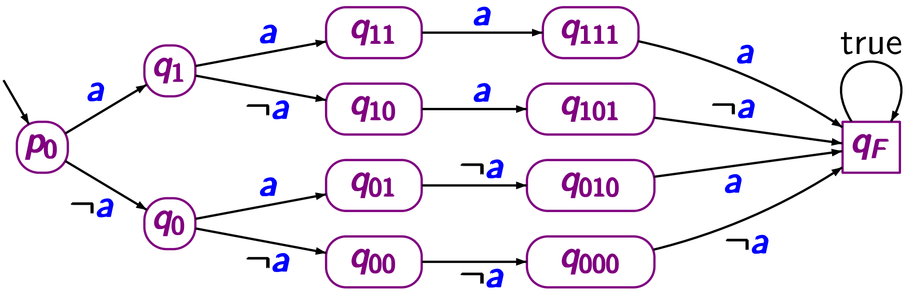
LT property E_n for n=2 and $AP = \{a\}$



general case: each **NBA** for E_n has $\geq 2^n$ states

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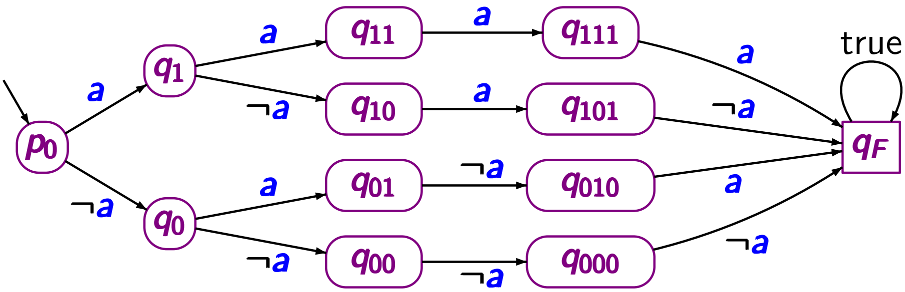

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$$E_n = Words(\varphi_n) \text{ where } \varphi_n = \bigwedge_{a \in AP} \bigwedge_{0 \leq i < n} (\bigcirc^i a \leftrightarrow \bigcirc^{n+i} a)$$

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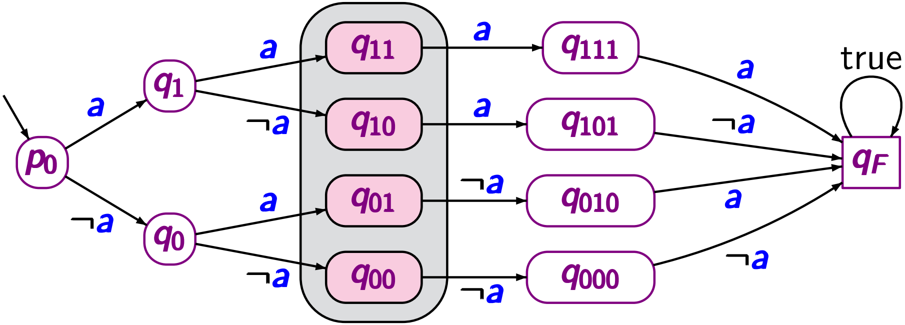

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