Compiler Construction

Lecture 8: Syntax Analysis IV ($LR(k)$ Grammars)

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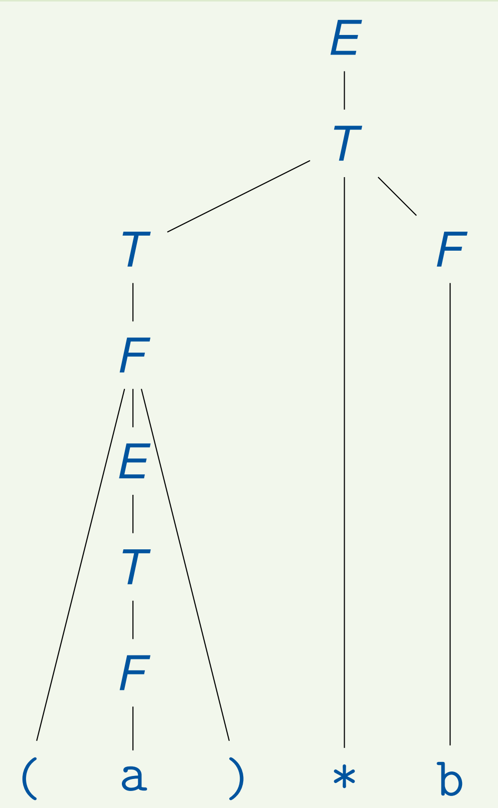
Recap: Top-Down Parsing

Example 8.1

Grammar for
arithmetic expressions:

$$\begin{array}{lllr} G_{AE}: & E \rightarrow E{+}T \mid T & & (1,2) \\ & T \rightarrow T{*}F \mid F & & (3,4) \\ & F \rightarrow (E) \mid \mathtt{a} \mid \mathtt{b} & & (5,6,7) \end{array}$$

Leftmost analysis of `(a)*b`:
2 3 4 5 2 4 6 7



Bottom-Up Parsing I

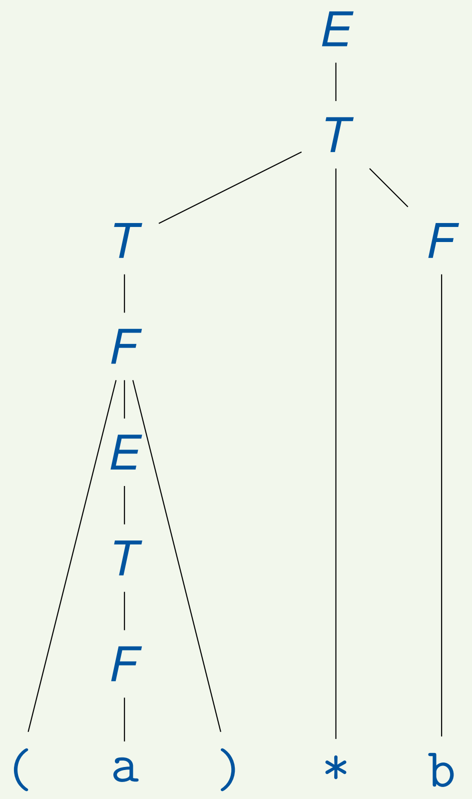
Example 8.2

Grammar for arithmetic expressions:

$$
\begin{aligned}
G_{AE}:\ & E \to E{+}T \mid T && (1,2)\\
& T \to T{*}F \mid F && (3,4)\\
& F \to (E) \mid \mathtt{a} \mid \mathtt{b} && (5,6,7)
\end{aligned}
$$

Reversed rightmost analysis of (a)*b:

6 4 2 5 4 7 3 2



Bottom-Up Parsing II

Approach:

1. Given $G \in CFG_\Sigma$, construct a nondeterministic bottom-up parsing automaton (NBA) which accepts $L(G)$ and which additionally computes corresponding (reversed) rightmost analyses
   – input alphabet: Σ
   – pushdown alphabet: X
   – output alphabet: $[p]$ (where $p := |P|$)
   – state set: omitted
   – transitions:
     shift: shifting input symbols onto the pushdown
     reduce: replacing the right-hand side of a production by its left-hand side (= inverse expansion step)
2. Remove nondeterminism by allowing lookahead on the input:
   $G \in LR(k)$ iff $L(G)$ recognisable by deterministic bottom-up parsing automaton with lookahead of k symbols

Nondeterministic Bottom-Up Automaton I

Definition 8.3 (Nondeterministic bottom-up parsing automaton)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$. The nondeterministic bottom-up parsing automaton of G, $NBA(G)$, is defined by the following components.

- Input alphabet: Σ
- Pushdown alphabet: X
- Output alphabet: $[p]$
- Configurations: $\Sigma^* \times X^* \times [p]^*$ (top of pushdown to the right)
- Transitions for $w \in \Sigma^*$, $\alpha \in X^*$, and $z \in [p]^*$:
  shifting steps: $(aw, \alpha, z) \vdash (w, \alpha a, z)$ if $a \in \Sigma$
  reduction steps: $(w, \alpha\beta, z) \vdash (w, \alpha A, zi)$ if $\pi_i = A \to \beta$
- Initial configuration for $w \in \Sigma^*$: $(w, \varepsilon, \varepsilon)$
- Final configurations: $\{\varepsilon\} \times \{S\} \times [p]^*$

Nondeterministic Bottom-Up Automaton II

Example 8.4

Grammar for arithmetic expressions (cf. Example 8.2):

$$
\begin{aligned}
G_{AE} : E &\rightarrow E{+}T \mid T && (1,2)\\
T &\rightarrow T{*}F \mid F && (3,4)\\
F &\rightarrow (E) \mid a \mid b && (5,6,7)
\end{aligned}
$$

Bottom-up parsing of $(a)*b$:

$$
\begin{aligned}
& ((a)*b, \varepsilon, \varepsilon)\\
\vdash\ & (a)*b, (, \varepsilon)\\
\vdash\ & ()*b, (a, \varepsilon)\\
\vdash\ & ()*b, (F, 6)\\
\vdash\ & ()*b, (T, 64)\\
\vdash\ & ()*b, (E, 642)\\
\vdash\ & (*b, (E), 642)\\
\vdash\ & (*b, F, 6425)\\
\vdash\ & (*b, T, 64254)\\
\vdash\ & (b, T*, 64254)\\
\vdash\ & (\varepsilon, T*b, 64254)\\
\vdash\ & (\varepsilon, T*F, 642547)\\
\vdash\ & (\varepsilon, T, 6425473)\\
\vdash\ & (\varepsilon, E, 64254732)
\end{aligned}
$$

Correctness of $NBA(G)$

Theorem 8.5 (Correctness of $NBA(G)$)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ and $NBA(G)$ as before. Then, for every $w \in \Sigma^*$ and $z \in [p]^*$,

$$(w, \varepsilon, \varepsilon) \vdash^* (\varepsilon, S, z) \quad \text{iff} \quad \overleftarrow{z} \text{ is a rightmost analysis of } w$$

Proof.

similar to the top-down case (Theorem 6.1) □

Nondeterministic Bottom-Up Parsing

Nondeterminisn in $NBA(G)$

Observation: $NBA(G)$ is generally nondeterministic

- Shift or reduce? Example:

$$(bw, \alpha a, z) \vdash \begin{cases} (w, \alpha ab, z) \\ (bw, \alpha A, zi) \end{cases} \text{if } \pi_i = A \to a$$

- If reduce: which "handle" β? Example:

$$(w, \alpha ab, z) \vdash \begin{cases} (w, \alpha A, zi) \\ (w, \alpha aB, zj) \end{cases} \text{if } \pi_i = A \to ab \text{ and } \pi_j = B \to b$$

- If reduce β: which left-hand side A? Example:

$$(w, \alpha a, z) \vdash \begin{cases} (w, \alpha A, zi) \\ (w, \alpha B, zj) \end{cases} \text{if } \pi_i = A \to a \text{ and } \pi_j = B \to a$$

- When to terminate parsing? Example:

$$\underbrace{(\varepsilon, S, z)}_{\text{final}} \vdash (\varepsilon, A, zi) \text{ if } \pi_i = A \to S$$

Resolving Termination Nondeterminism I

General assumption to avoid nondeterminism of last type:
every grammar is start separated

Definition 8.6 (Start separation)

A grammar $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ is called start separated if S only occurs in productions of the form $S \to A$ where $A \neq S$.

Remarks:

- Start separation always possible by adding $S' \to S$ with new start symbol S'
- From now on consider only reduced grammars of this form (and let $\pi_0 := S' \to S$)

Resolving Termination Nondeterminism II

Start separation removes "When to terminate parsing?" nondeterminism:

Lemma 8.7

If $G \in CFG_\Sigma$ is start separated, then no successor of a final configuration (ε, S', z) in $NBA(G)$ is again a final configuration.
(Thus parsing should be stopped in the first final configuration.)

Proof.

- To (ε, S', z), only reductions by ε-productions can be applied:

$$(\varepsilon, S', z) \vdash (\varepsilon, S'A, zi) \quad \text{if } \pi_i = A \to \varepsilon$$

- Thereafter, only reductions by productions of the form $A_0 \to A_1 \ldots A_n$ ($n \geq 0$) applicable
- Every resulting configuration is of the (non-final) form

$$(\varepsilon, S'B_1 \ldots B_k, z) \quad \text{where } k \geq 1$$

□

$LR(k)$ Grammars I

Goal: resolve remaining nondeterminism of $NBA(G)$ by supporting lookahead of $k \in \mathbb{N}$ symbols on the input

$\Longrightarrow$ $LR(k)$: reading of input from left to right with k-lookahead, computing a rightmost analysis

Definition 8.8 ($LR(k)$ grammar)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ be start separated and $k \in \mathbb{N}$. Then G has the $LR(k)$ property (notation: $G \in LR(k)$) if for all rightmost derivations of the form

$$S \begin{cases} \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w \\ \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y \end{cases}$$

such that $\mathrm{first}_k(w) = \mathrm{first}_k(y)$, it follows that $\alpha = \gamma$, $A = B$, and $x = y$.

$LR(k)$ Grammars II

Remarks:

- If $G \in LR(k)$, then the reduction of $\alpha\beta w$ to $\alpha A w$ is already determined by $\mathrm{first}_k(w)$.
- Therefore $NBA(G)$ in configuration $(w, \alpha\beta, z)$ can decide to reduce and how to reduce.
- Computation of $NBA(G)$ for $S \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w$:

$$(w'w, \varepsilon, \varepsilon) \vdash^* (w, \alpha\beta, z) \overset{\text{red } i}{\vdash} (w, \alpha A, zi) \vdash \ldots$$

  where $\pi_i = A \to \beta$
- Computation of $NBA(G)$ for $S \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y$:
  - with direct reduction ($y = x$, $\alpha\beta = \gamma\delta$, $\pi_j = B \to \delta$):

$$(y'y, \varepsilon, \varepsilon) \vdash^* (y, \alpha\beta, z') = (x, \gamma\delta, z') \overset{\text{red } j}{\vdash} (x, \gamma B, z'j) \vdash \ldots$$

  - with previous shifts ($y = x'x$, $\alpha\beta x' = \gamma\delta$, $\pi_j = B \to \delta$):

$$(y'y, \varepsilon, \varepsilon) \vdash^* (y, \alpha\beta, z') = (x'x, \alpha\beta, z') \overset{\text{shift}^*}{\vdash} (x, \alpha\beta x', z') = (x, \gamma\delta, z') \overset{\text{red } j}{\vdash} (x, \gamma B, z'j) \vdash \ldots$$

$LR(0)$ Grammars

The case $k = 0$ is relevant (in contrast to $LL(0)$): here the decision is just based on the contents of the pushdown, without any lookahead.

Corollary 8.9 ($LR(0)$ grammar)

$G \in CFG_\Sigma$ has the $LR(0)$ property if for all rightmost derivations of the form

$$S \begin{cases} \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w \\ \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y \end{cases}$$

it follows that $\alpha = \gamma$, $A = B$, and $x = y$.

Goal: derive a finite information from the pushdown which suffices to resolve the nondeterminism (similar to abstraction of right context in LL parsing by fo-sets)

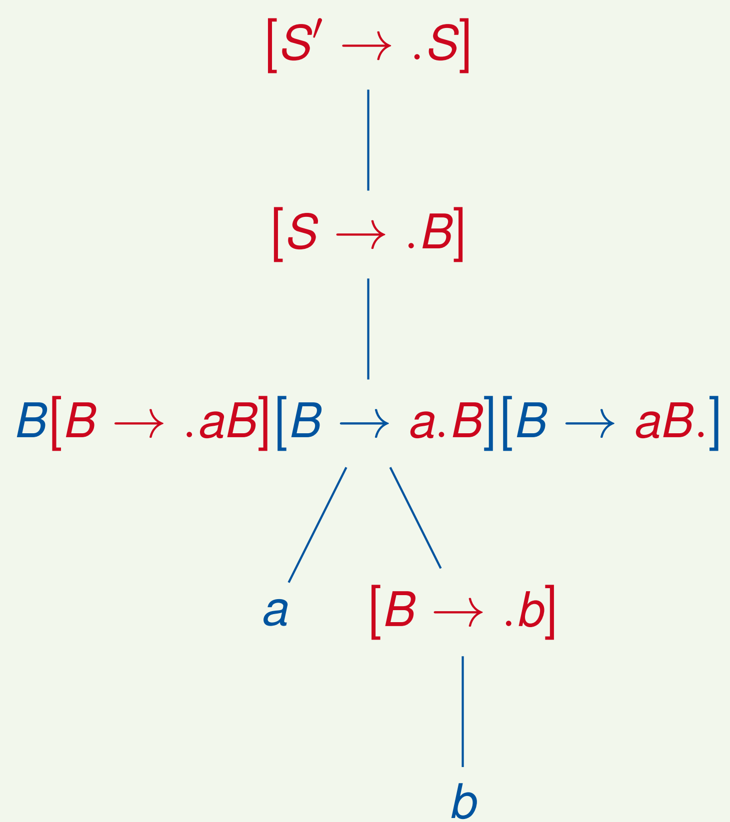
$LR(0)$ Items and Sets I

Example 8.10

$$
\begin{aligned}
G:\ S' &\to S & (0)\\
S &\to B \mid C & (1,2)\\
B &\to aB \mid b & (3,4)\\
C &\to aC \mid c & (5,6)
\end{aligned}
$$

$NBA(G)$:

$$
\begin{aligned}
& (ab, \varepsilon, \varepsilon)\\
\vdash\ & (b, a, \varepsilon)\\
\vdash\ & (\varepsilon, ab, \varepsilon)\\
\vdash\ & (\varepsilon, aB, 4)\\
\vdash\ & (\varepsilon, B, 43)\\
\vdash\ & (\varepsilon, S, 431)\\
\vdash\ & (\varepsilon, S', 4310)
\end{aligned}
$$



$LR(0)$ Items and Sets II

Definition 8.11 ($LR(0)$ items and sets)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ be start separated by $S' \to S$ and $S' \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta_1\beta_2 w$ (i.e., $A \to \beta_1\beta_2 \in P$).

- $[A \to \beta_1 \cdot \beta_2]$ is called an $LR(0)$ item for $\alpha\beta_1$.
- Given $\gamma \in X^*$, $LR(0)(\gamma)$ denotes the set of all $LR(0)$ items for γ, called the $LR(0)$ set (or: $LR(0)$ information) of γ.
- $LR(0)(G) := \{LR(0)(\gamma) \mid \gamma \in X^*\}$.

Corollary 8.12

1. *For every $\gamma \in X^*$, $LR(0)(\gamma)$ is finite.*
2. *$LR(0)(G)$ is finite.*
3. *The item $[A \to \beta\cdot] \in LR(0)(\gamma)$ indicates the possible reduction $(w, \alpha\beta, z) \vdash (w, \alpha A, zi)$ where $\pi_i = A \to \beta$ and $\gamma = \alpha\beta$.*
4. *The item $[A \to \beta_1 \cdot Y\beta_2] \in LR(0)(\gamma)$ indicates an incomplete handle β_1 (to be completed by shift operations or ε-reductions).*

$LR(0)$ Conflicts

Definition 8.13 ($LR(0)$ conflicts)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ and $I \in LR(0)(G)$.

- I has a shift/reduce conflict if there exist $A \to \alpha_1 a \alpha_2, B \to \beta \in P$ such that

$$[A \to \alpha_1 \cdot a\alpha_2], [B \to \beta \cdot] \in I.$$

- I has a reduce/reduce conflict if there exist $A \to \alpha, B \to \beta \in P$ with $A \neq B$ or $\alpha \neq \beta$ such that

$$[A \to \alpha \cdot], [B \to \beta \cdot] \in I.$$

Lemma 8.14

$G \in LR(0)$ iff no $I \in LR(0)(G)$ contains conflicting items.

Proof.

omitted □

Computing $LR(0)$ Sets I

Theorem 8.15 (Computing $LR(0)$ sets)

Let $G = \langle N, \Sigma, P, S\rangle \in CFG_\Sigma$ be start separated by $S' \to S$ and reduced.

1. $LR(0)(\varepsilon)$ *is the least set such that*
   - $[S' \to \cdot S] \in LR(0)(\varepsilon)$ *and*
   - *if* $[A \to \cdot B\gamma] \in LR(0)(\varepsilon)$ *and* $B \to \beta \in P$,
     then $[B \to \cdot\beta] \in LR(0)(\varepsilon)$.
2. $LR(0)(\alpha Y)$ *(*$\alpha \in X^*$, $Y \in X$*) is the least set such that*
   - *if* $[A \to \gamma_1 \cdot Y\gamma_2] \in LR(0)(\alpha)$,
     then $[A \to \gamma_1 Y \cdot \gamma_2] \in LR(0)(\alpha Y)$ *and*
   - *if* $[A \to \gamma_1 \cdot B\gamma_2] \in LR(0)(\alpha Y)$ *and* $B \to \beta \in P$,
     then $[B \to \cdot\beta] \in LR(0)(\alpha Y)$.

Computing $LR(0)$ Sets II

Example 8.16 (cf. Example 8.10)

$G: \; S' \to S \qquad S \to B \mid C$

$\quad\;\; B \to aB \mid b \quad C \to aC \mid c$

$[S' \to \cdot S] \in$

$LR(0)(\varepsilon)$ $[A \to \cdot B\gamma] \in LR(0)(\varepsilon), B \to \beta \in P \implies [B \to \cdot\beta] \in LR(0)(\varepsilon)$ $\quad [A \to \gamma_1 \cdot Y\gamma_2] \in LR(0)(\alpha) \implies [A \to \gamma_1 Y \cdot \gamma_2] \in LR(0)(\alpha Y)$ $\quad [A \to \gamma_1 \cdot B\gamma_2] \in$ $\implies [B \to \cdot\beta] \in$

$I_0 := LR(0)(\varepsilon)$: $[S' \to \cdot S]$ $[S \to \cdot B]$ $[S \to \cdot C]$ $[B \to \cdot aB]$ $[B \to \cdot b]$ $[C \to \cdot aC]$ $[C \to \cdot c]$

$I_1 := LR(0)(S)$: $[S' \to S \cdot]$

$I_2 := LR(0)(B)$: $[S \to B \cdot]$

$I_3 := LR(0)(C)$: $[S \to C \cdot]$

$I_4 := LR(0)(a)$: $[B \to a \cdot B]$ $[C \to a \cdot C]$ $[B \to \cdot aB]$ $[B \to \cdot b]$ $[C \to \cdot aC]$ $[C \to \cdot c]$

$I_5 := LR(0)(b)$: $[B \to b \cdot]$

$I_6 := LR(0)(c)$: $[C \to c \cdot]$

$I_7 := LR(0)(aB)$: $[B \to aB \cdot]$

$I_8 := LR(0)(aC)$: $[C \to aC \cdot]$

($LR(0)(aa) = LR(0)(a) = I_4$, $LR(0)(ab) = LR(0)(b) = I_5$, $LR(0)(ac) = LR(0)(c) = I_6$, ...,

$I_9 := LR(0)(\gamma) = \emptyset$ in all remaining cases, e.g., for $\gamma = bB$)

No conflicts $\implies G \in LR(0)$ (but $G \notin LL(1)$)

Examples of $LR(0)$ Conflicts

Shift/Reduce Conflicts

Example 8.17

$$G: \quad S' \to S$$
$$\qquad S \to aS \mid a$$

$$\begin{array}{lllll}
LR(0)(\varepsilon): & [S' \to \cdot S] & [S \to \cdot aS] & [S \to \cdot a] & \\
LR(0)(S): & [S' \to S\cdot] & & & \\
LR(0)(a): & [S \to a \cdot S] & [S \to \cdot aS] & [S \to \cdot a] & [S \to a\cdot] \\
LR(0)(aS): & [S \to aS\cdot] & & &
\end{array}$$

Note: G is unambiguous

Examples of $LR(0)$ Conflicts

Reduce/Reduce Conflicts

Example 8.18

$$G: \quad S' \to S$$
$$S \to Aa \mid Bb$$
$$A \to a$$
$$B \to a$$

$LR(0)(\varepsilon)$: $[S' \to \cdot S]$ $[S \to \cdot Aa]$ $[S \to \cdot Bb]$ $[A \to \cdot a]$ $[B \to \cdot a]$
$LR(0)(S)$: $[S' \to S\cdot]$
$LR(0)(A)$: $[S \to A \cdot a]$
$LR(0)(B)$: $[S \to B \cdot b]$
$LR(0)(a)$: $[A \to a\cdot]$ $[B \to a\cdot]$
$LR(0)(Aa)$: $[S \to Aa\cdot]$
$LR(0)(Bb)$: $[S \to Bb\cdot]$

Note: G is unambiguous