



Compiler Construction

Lecture 5: Syntax Analysis I (Introduction)

Summer Semester 2017

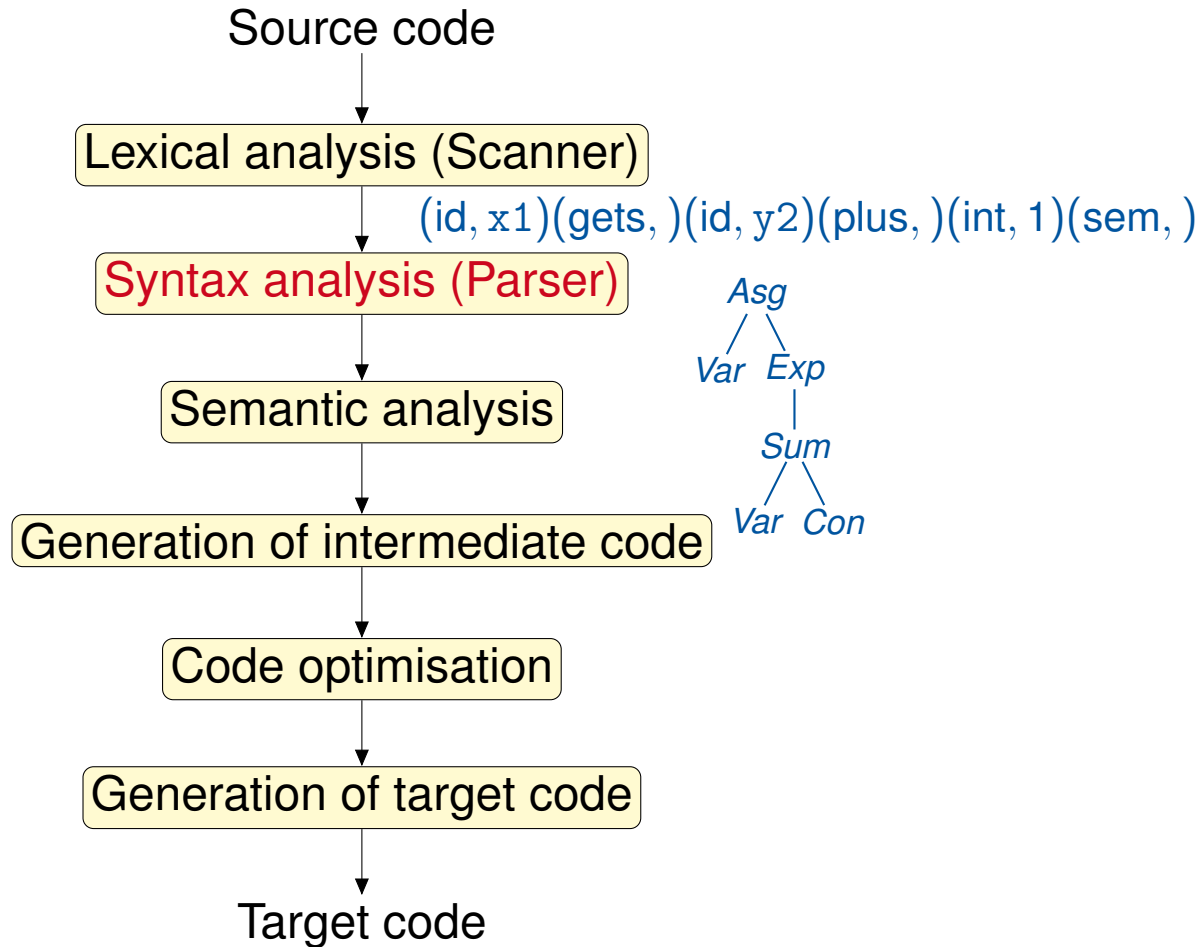
Thomas Noll

Software Modeling and Verification Group

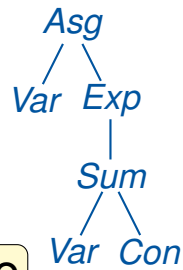
RWTH Aachen University

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Conceptual Structure of a Compiler



context-free grammars/
pushdown automata



Problem Statement

Outline of Lecture 5

Problem Statement

Context-Free Grammars and Languages

Parsing Context-Free Languages

Nondeterministic Top-Down Parsing

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Syntactic Structures

From Merriam-Webster's Online Dictionary

Syntax: the way in which linguistic elements (as words) are put together to form constituents (as phrases or clauses)

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- **Starting point:** sequence of symbols as produced by the scanner
 - here: ignore attribute information
 - Σ (finite) set of **tokens** (= syntactic atoms/**terminal symbols**, (e.g., {id, if, int, ...}))
 - $w \in \Sigma^*$ **token sequence** (obviously, not every $w \in \Sigma^*$ forms a valid program)

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- **Syntactic units:**
 - atomic:** keywords, variable/type/procedure/... identifiers, numerals, arithmetic/Boolean operators, ...
 - complex:** declarations, arithmetic/Boolean expressions, statements, ...
- **Observation:** the hierarchical structure of (complex) syntactic units can be described by **context-free grammars**

Problem Statement

Syntax Analysis

Definition 5.1

The goal of **syntax analysis** is to determine the syntactic structure of a program, given by a token sequence, according to a context-free grammar.

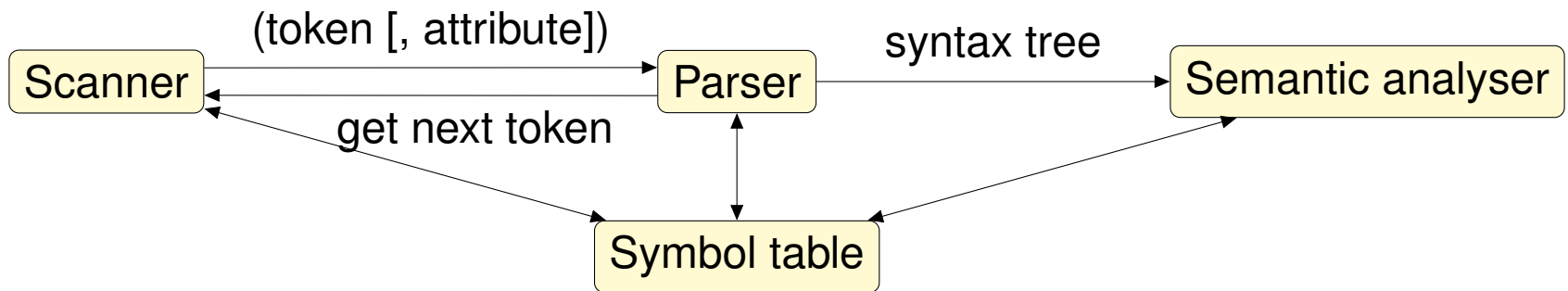
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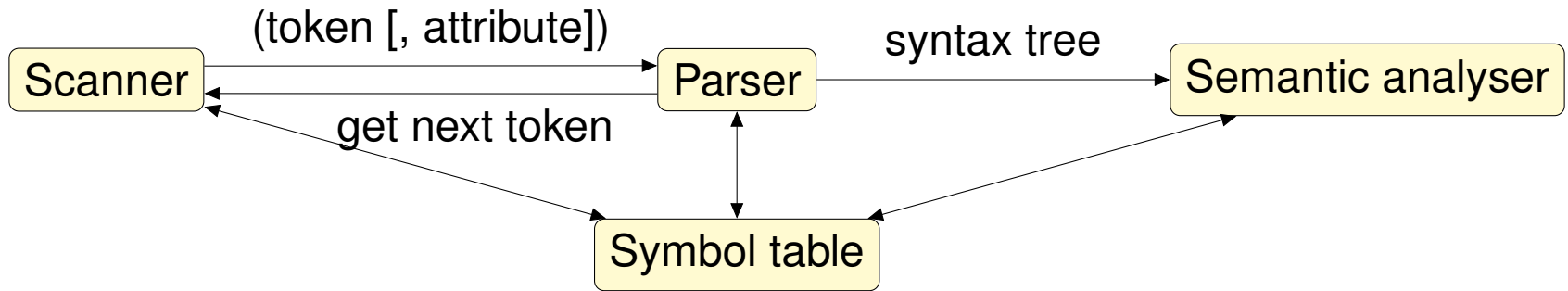
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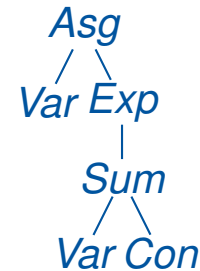
Example:

... $x_1 := y_2 + 1; \dots$

↓ Scanner

... (id, p_1)(gets,)(id, p_2)(plus,)(int, 1)(sem,) ...

Parser →



Context-Free Grammars and Languages

Outline of Lecture 5

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Context-Free Grammars and Languages

Context-Free Grammars I

Definition 5.2 (Syntax of context-free grammars)

A **context-free grammar (CFG)** (over Σ) is a quadruple $G = \langle N, \Sigma, P, S \rangle$ where

- N is a finite set of **nonterminal symbols**,
- Σ is a (finite) alphabet of **terminal symbols** (disjoint from N),
- P is a finite set of **production rules** of the form $A \rightarrow \alpha$ where
 - $A \in N$ and
 - $\alpha \in X^*$ for $X := N \cup \Sigma$,
- $S \in N$ is a **start symbol**.

The set of all context-free grammars over Σ is denoted by CFG_{Σ} .

Context-Free Grammars and Languages

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Remarks: as denotations we generally use

- $A, B, C, \dots \in N$ for nonterminal symbols
- $a, b, c, \dots \in \Sigma$ for terminal symbols
- $u, v, w, x, y, \dots \in \Sigma^*$ for terminal words
- $\alpha, \beta, \gamma, \dots \in X^*$ for **sentences**

Context-Free Grammars and Languages

Context-Free Grammars II

Context-free grammars generate context-free languages:

Definition 5.3 (Semantics of context-free grammars)

Let $G = \langle N, \Sigma, P, S \rangle$ be a context-free grammar.

- The **derivation relation** $\Rightarrow \subseteq X^+ \times X^*$ of G is defined by

$\alpha \Rightarrow \beta$ iff there exist $\alpha_1, \alpha_2 \in X^*$, $A \rightarrow \gamma \in P$ such that $\alpha = \alpha_1 A \alpha_2$ and $\beta = \alpha_1 \gamma \alpha_2$.

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Remark: obviously, $L(G) = \{w \in \Sigma^* \mid S \Rightarrow_l^* w\} = \{w \in \Sigma^* \mid S \Rightarrow_r^* w\}$

Context-Free Grammars and Languages

Context-Free Languages

Example 5.4

The grammar $G = \langle N, \Sigma, P, S \rangle \in CFG_{\Sigma}$ over $\Sigma := \{a, b\}$, given by the productions

$$S \rightarrow aSb \mid \varepsilon,$$

generates the context-free (and non-regular) language

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Context-Free Grammars and Languages

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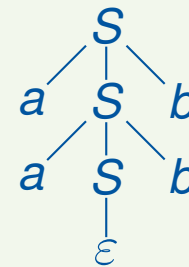
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The example derivation

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

can be represented by the following **syntax tree** for *aabb*:



Syntax Trees, Derivations, and Words

Observations

1. Every syntax tree yields exactly one word (= concatenation of terminal leaves).
2. Every syntax tree corresponds to exactly one leftmost derivation, and vice versa.
3. Every syntax tree corresponds to exactly one rightmost derivation, and vice versa.

Thus: syntax trees are **uniquely** representable by leftmost/rightmost derivations.

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Thus: syntax trees are **uniquely** representable by leftmost/rightmost derivations.

But: a word can have **several** syntax trees (see next slide)

Ambiguity of CFGs and CFLs I

Definition 5.5 (Ambiguity)

- A context-free grammar $G \in CFG_{\Sigma}$ is called **unambiguous** if every word $w \in L(G)$ has exactly one syntax tree. Otherwise it is called **ambiguous**.

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Context-Free Grammars and Languages

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on the board

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Example 5.6

on the board

Corollary 5.7

*A grammar $G \in CFG_{\Sigma}$ is unambiguous
iff every word $w \in L(G)$ has exactly one leftmost derivation
iff every word $w \in L(G)$ has exactly one rightmost derivation.*

Ambiguity of CFGs and CFLs II

Theorem 5.8

*It is generally **undecidable** whether a given CFG is ambiguous or not.*

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Proof (idea).

Reduction from **Post Correspondence Problem**: given instance (\vec{x}, \vec{y}) of PCP, construct CFG G with two “branches” $S \rightarrow X \mid Y$ that respectively enumerate all \vec{x}/\vec{y} -concatenations (plus some index information).

Result: G is ambiguous iff (\vec{x}, \vec{y}) has a solution
(see [Hopcroft/Motwani/Ullman 2001, Section 9.5.2] for details) □

Context-Free Grammars and Languages

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Remark: *resolution* of ambiguities by parser (later):

- yacc: operator precedences and associativities
- ANTLR: predicates

Parsing Context-Free Languages

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The Word Problem for Context-Free Languages

Problem 5.9 (Word problem for context-free languages)

Given $G \in CFG_{\Sigma}$ and $w \in \Sigma^$, decide whether $w \in L(G)$ (and determine a corresponding syntax tree).*

Parsing Context-Free Languages

The Word Problem for Context-Free Languages

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Given $G \in CFG_{\Sigma}$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ (and determine a corresponding syntax tree).

This problem is **decidable** for arbitrary CFGs:

- (for CFGs in Chomsky Normal Form)
Using the **tabular method by Cocke, Younger, and Kasami** (“CYK Algorithm”; time/space complexity $\mathcal{O}(|w|^3)/\mathcal{O}(|w|^2)$)

Parsing Context-Free Languages

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- Using the **predecessor method**:

$$w \in L(G) \iff S \in \text{pre}^*(\{w\})$$

where $\text{pre}^*(M) := \{\alpha \in X^* \mid \alpha \Rightarrow^* \beta \text{ for some } \beta \in M\}$
(polynomial [non-linear] time complexity)

Parsing Context-Free Languages

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Goal: exploit the special syntactic structures as present in programming languages (usually: no ambiguities) to devise parsing methods which are based on **deterministic pushdown automata with linear space and time complexity**

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Two approaches:

Top-down parsing: construction of syntax tree from the **root towards the leaves**, representation as **leftmost derivation**

Bottom-up parsing: construction of syntax tree from the **leaves towards the root**, representation as (reversed) **rightmost derivation**

Parsing Context-Free Languages

Leftmost/Rightmost Analysis I

Goal: compact representation of left-/rightmost derivations by index sequences

Definition 5.10 (Leftmost/rightmost analysis)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_{\Sigma}$ where $P = \{\pi_1, \dots, \pi_p\}$.

- If $i \in [p]$, $\pi_i = A \rightarrow \gamma$, $w \in \Sigma^*$, and $\alpha \in X^*$, then we write

$$wA\alpha \xRightarrow{i}_l w\gamma\alpha \quad \text{and} \quad \alpha Aw \xRightarrow{i}_r \alpha\gamma w.$$

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- An index sequence $z \in [p]^*$ is called a **leftmost analysis** (**rightmost analysis**) of α if $S \xRightarrow{z}_l \alpha$ ($S \xRightarrow{z}_r \alpha$), respectively.

Leftmost/Rightmost Analysis II

Example 5.11

Grammar for arithmetic expressions: $G_{AE} : E \rightarrow E+T \mid T \quad (1, 2)$
 $T \rightarrow T*F \mid F \quad (3, 4)$
 $F \rightarrow (E) \mid a \mid b \quad (5, 6, 7)$

Leftmost/Rightmost Analysis II

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Leftmost derivation of $(a)*b$:

$$\begin{array}{ccccccc} E & \xRightarrow{2}_l & T & \xRightarrow{3}_l & T*F & \xRightarrow{4}_l & F*F & \xRightarrow{5}_l & (E)*F \\ & \xRightarrow{2}_l & (T)*F & \xRightarrow{4}_l & (F)*F & \xRightarrow{6}_l & (a)*F & \xRightarrow{7}_l & (a)*b \end{array}$$

Parsing Context-Free Languages

Leftmost/Rightmost Analysis II

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Parsing Context-Free Languages

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Rightmost derivation of $(a)*b$:

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Leftmost/Rightmost Analysis II

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\Rightarrow rightmost analysis: 23745246

Parsing Context-Free Languages

Reducedness of Context-Free Grammars

General assumption in the following: every grammar is reduced

Definition 5.12 (Reduced CFG)

A grammar $G = \langle N, \Sigma, P, S \rangle \in \text{CFG}_\Sigma$ is called **reduced** if for every $A \in N$ there exist $\alpha, \beta \in X^*$ and $w \in \Sigma^*$ such that

$$S \Rightarrow^* \alpha A \beta \quad (A \text{ reachable}) \text{ and}$$

$$A \Rightarrow^* w \quad (A \text{ productive}).$$

Nondeterministic Top-Down Parsing

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Top-Down Parsing

Approach:

1. Given $G \in CFG_{\Sigma}$, construct a **nondeterministic pushdown automaton** (PDA) which accepts $L(G)$ and which additionally computes corresponding leftmost derivations (similar to the proof of “ $L(CFG_{\Sigma}) \subseteq L(PDA_{\Sigma})$ ”)
 - input alphabet: Σ
 - pushdown alphabet: X
 - output alphabet: $[p]$
 - state set: not required

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 - input alphabet: Σ
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 - output alphabet: $[p]$
 - state set: not required
2. **Remove nondeterminism** by supporting **lookahead** on the input:
 $G \in LL(k)$ iff $L(G)$ recognisable by deterministic PDA with lookahead of k symbols

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton I

Definition 5.13 (Nondeterministic top-down parsing automaton)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_{\Sigma}$. The **nondeterministic top-down parsing automaton** of G , $NTA(G)$, is defined by the following components.

- **Input alphabet:** Σ
- **Pushdown alphabet:** X
- **Output alphabet:** $[p]$
- **Configurations:** $\Sigma^* \times X^* \times [p]^*$ (top of pushdown to the left)
- **Transitions** for $w \in \Sigma^*$, $\alpha \in X^*$, and $z \in [p]^*$:
 - expansion steps: if $\pi_i = A \rightarrow \beta$, then $(w, A\alpha, z) \vdash (w, \beta\alpha, zi)$
 - matching steps: for every $a \in \Sigma$, $(aw, a\alpha, z) \vdash (w, \alpha, z)$
- **Initial configuration** for $w \in \Sigma^*$: (w, S, ε)
- **Final configurations:** $\{\varepsilon\} \times \{\varepsilon\} \times [p]^*$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton I

Definition 5.13 (Nondeterministic top-down parsing automaton)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_{\Sigma}$. The **nondeterministic top-down parsing automaton** of G , $NTA(G)$, is defined by the following components.

- **Input alphabet:** Σ
- **Pushdown alphabet:** X
- **Output alphabet:** $[p]$
- **Configurations:** $\Sigma^* \times X^* \times [p]^*$ (top of pushdown to the left)
- **Transitions** for $w \in \Sigma^*$, $\alpha \in X^*$, and $z \in [p]^*$:
 - expansion steps: if $\pi_i = A \rightarrow \beta$, then $(w, A\alpha, z) \vdash (w, \beta\alpha, zi)$
 - matching steps: for every $a \in \Sigma$, $(aw, a\alpha, z) \vdash (w, \alpha, z)$
- **Initial configuration** for $w \in \Sigma^*$: (w, S, ε)
- **Final configurations:** $\{\varepsilon\} \times \{\varepsilon\} \times [p]^*$

Remark: $NTA(G)$ is nondeterministic iff G contains $A \rightarrow \beta \mid \gamma$

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$((a)*b, E, \varepsilon)$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \epsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \epsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned}G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7)\end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned}&((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345)\end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &((a)*b, E)*F, 2345) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &((a)*b, E)*F, 2345) \\ \vdash &((a)*b, T)*F, 23452) \\ \vdash &((a)*b, F)*F, 234524) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \\ \vdash &())*b,))*F, 2345246) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \\ \vdash &() *b,) *F, 2345246) \\ \vdash &(*b, *F, 2345246) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \\ \vdash &() *b,) *F, 2345246) \\ \vdash &(*b, *F, 2345246) \\ \vdash &(b, F, 2345246) \end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned}G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7)\end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned}&((a)*b, E, \epsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \\ \vdash &() *b,) *F, 2345246) \\ \vdash &(*b, *F, 2345246) \\ \vdash &(b, F, 2345246) \\ \vdash &(b, b, 23452467)\end{aligned}$$

Nondeterministic Top-Down Parsing

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

$$\begin{aligned} G_{AE} : E &\rightarrow E+T \mid T && (1, 2) \\ T &\rightarrow T*F \mid F && (3, 4) \\ F &\rightarrow (E) \mid a \mid b && (5, 6, 7) \end{aligned}$$

Leftmost analysis of $(a)*b$:

$$\begin{aligned} &((a)*b, E, \varepsilon) \\ \vdash &((a)*b, T, 2) \\ \vdash &((a)*b, T*F, 23) \\ \vdash &((a)*b, F*F, 234) \\ \vdash &((a)*b, (E)*F, 2345) \\ \vdash &(a)*b, E)*F, 2345) \\ \vdash &(a)*b, T)*F, 23452) \\ \vdash &(a)*b, F)*F, 234524) \\ \vdash &(a)*b, a)*F, 2345246) \\ \vdash &() *b,) *F, 2345246) \\ \vdash &(*b, *F, 2345246) \\ \vdash &(b, F, 2345246) \\ \vdash &(b, b, 23452467) \\ \vdash &(\varepsilon, \varepsilon, \mathbf{23452467}) \end{aligned}$$