Abstraction Refinement for Probabilistic Software

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6. Juli 2016

Outline

ANSI-C
program

Gambler's ruin

```
int main() {
1: int c = ndet(2) + 1;
2: while (0 < c && c < 3) {
3:   c += coin(0.5) ? 1:-1;
   }
4: assert(c > 0);
}
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Non-deterministic choice: **int** ndet(**int** n)

- No information about outcome
- Returns any value between 0 and n
- E.g. user input or underspecified function

Gambler's ruin

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Probabilistic choices: **int** coin(**float** p)

- Likelyhood of each possible outcome known
- Returns 1 with probability p and 0 with probability $1 - p$
- E.g. randomization or network communication

Gambler's ruin

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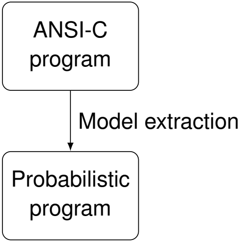
Outline



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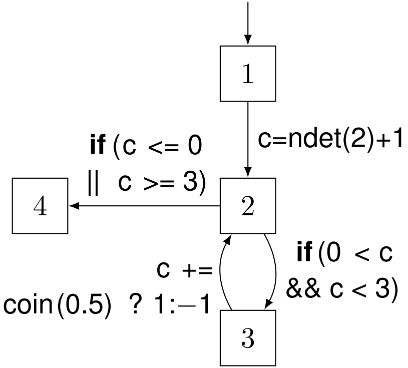
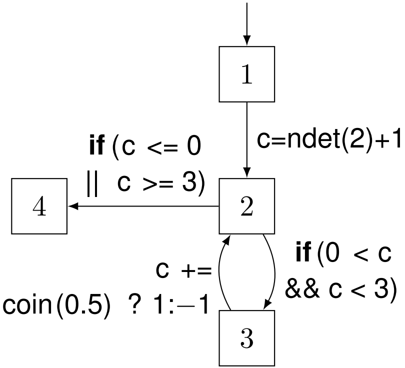
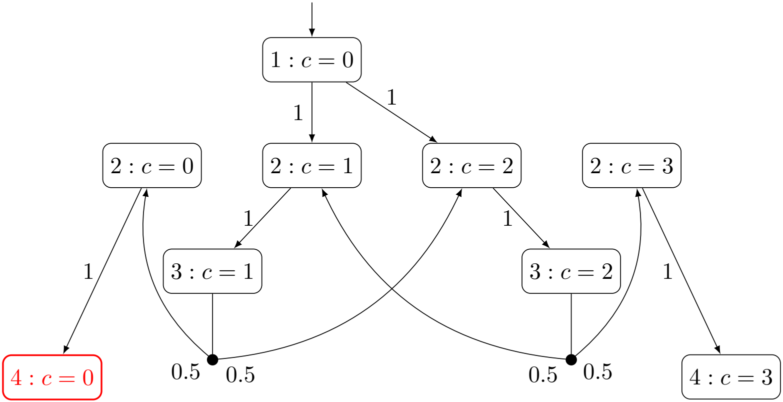


Semantics of probabilistic programs

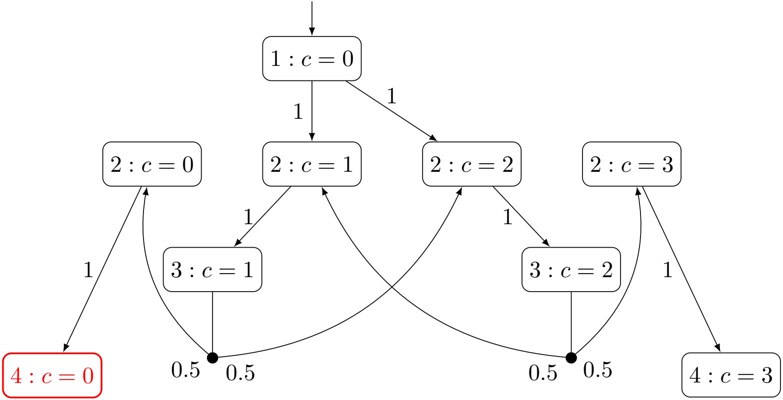


Semantics of probabilistic programs



Schedulers

- Schedulers resolve non-determinism by mapping each state to a non-deterministic choice



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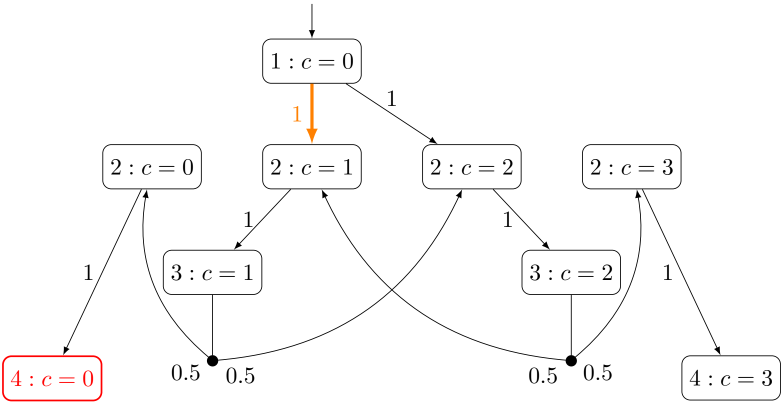

Schedulers

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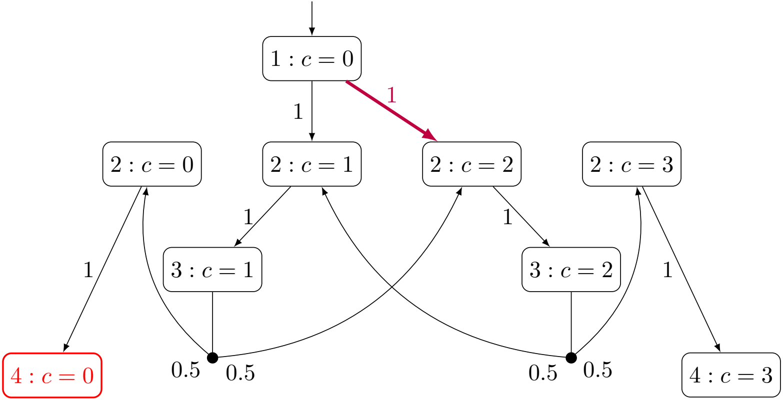

Probabilities

Problem

- Calculating *the* probability of a state is impossible
- Instead, there is a separate probability for each scheduler

Idea

- Define properties over minimal and maximal probabilities
- I.e. consider the *best-case* and the *worst-case* scheduler

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Probabilities

Probability of $\lozenge$red

- Scheduler $c = 1$: $\frac{1}{2}^1 + \frac{1}{2}^3 + \frac{1}{2}^5 + \cdots = \frac{2}{3}$

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Probabilities

Probability of $\Diamond$red

- Scheduler $c = 1$: $\frac{1}{2}^1 + \frac{1}{2}^3 + \frac{1}{2}^5 + \cdots = \frac{2}{3}$
- Scheduler $c = 2$: $\frac{1}{2} \times (\frac{1}{2}^1 + \frac{1}{2}^3 + \frac{1}{2}^5 + \dots) = \frac{1}{3}$

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Probabilities

Probability of $\Diamond$red

- Scheduler $c = 1$: $\frac{1}{2}^1 + \frac{1}{2}^3 + \frac{1}{2}^5 + \cdots = \frac{2}{3}$
  Maximal probability
- Scheduler $c = 2$: $\frac{1}{2} \times (\frac{1}{2}^1 + \frac{1}{2}^3 + \frac{1}{2}^5 + \ldots) = \frac{1}{3}$
  Minimal probability

Probabilities

Probability of $\lozenge$red

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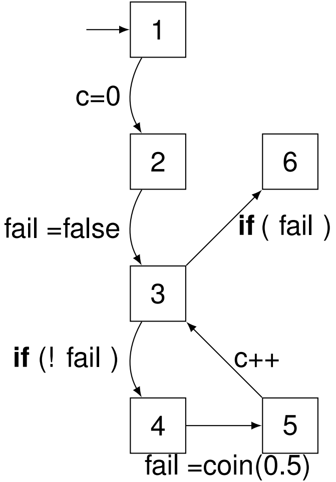
- Probabilities can be calculated using a linear optimisation problem

State space explosion

- MDP semantics often have too many (or even infinitely many) states

```
int main() {
1:   int c = 0;
2:   bool fail = false;
3:   while(!fail) {
4:     fail = coin(0.5);
5:     c++;
6:   }
}
```



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State space explosion

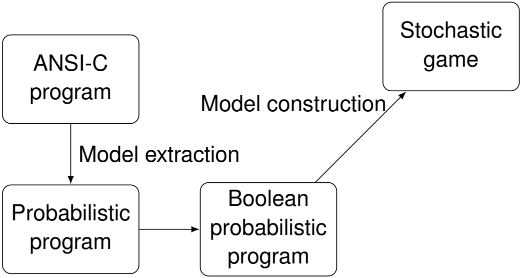
Problem

$$|\mathcal{M}| \approx \underbrace{2^{32} + \cdots + 2^{32}}_{\text{Number of 32-bit integer variables}} \times \text{Number of statements}$$

Idea

Reduce the value ranges of variables

Outline



ANSI-C program
Model extraction
Probabilistic program
Boolean probabilistic program
Model construction
Stochastic game

Predicate abstraction

Solution

Replace original variables with predicates (boolean expressions)

Predicate abstraction

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Replace original variables with predicates (boolean expressions)

```
int main() {
  int i = 0;
  while(true) {
    assert(i != 443);
    if(i < 10) {
      i = i + 1;
    }
    else i = i + 2;
  }
}
```

Predicate abstraction

Solution

Replace original variables with predicates (boolean expressions)

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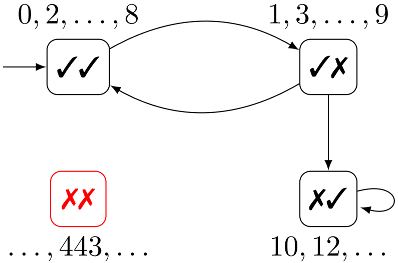
Predicates: $i < 10, i \bmod 2 = 0$



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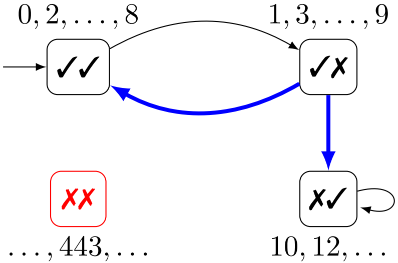

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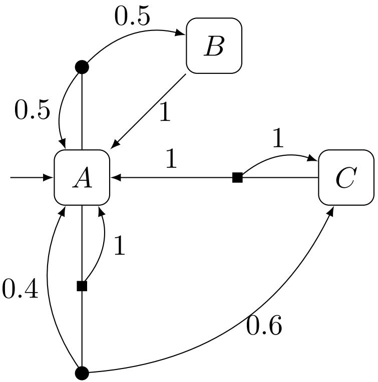
Two-player stochastic game

Definition

A two-player stochastic game is a tuple $\hat{\mathcal{M}} = (\hat{S}, \hat{s}_{init}, \hat{\mathbf{P}})$ with

- countable, non-empty set of states $\hat{S}$
- initial state $\hat{s}_{init} \in \hat{S}$
- transition function $\hat{\mathbf{P}} : \hat{S} \to \mathcal{P}(\mathcal{P}(Dist_{\hat{S}}))$



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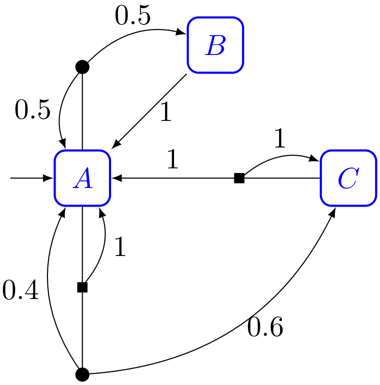

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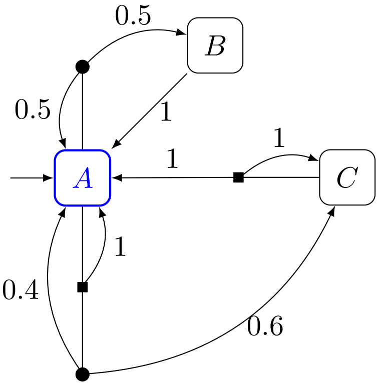

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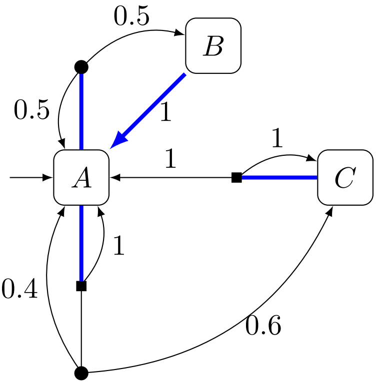

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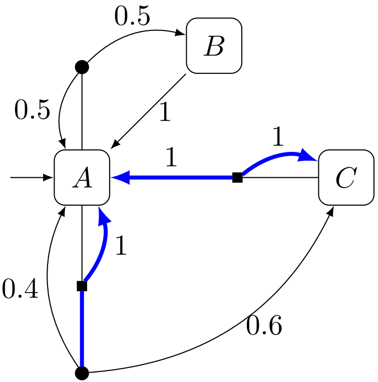

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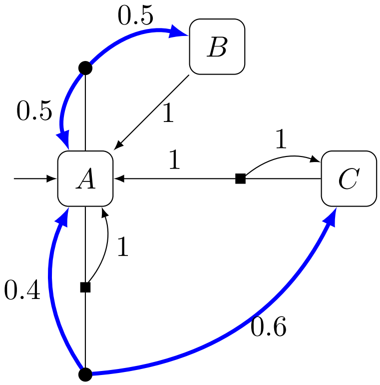
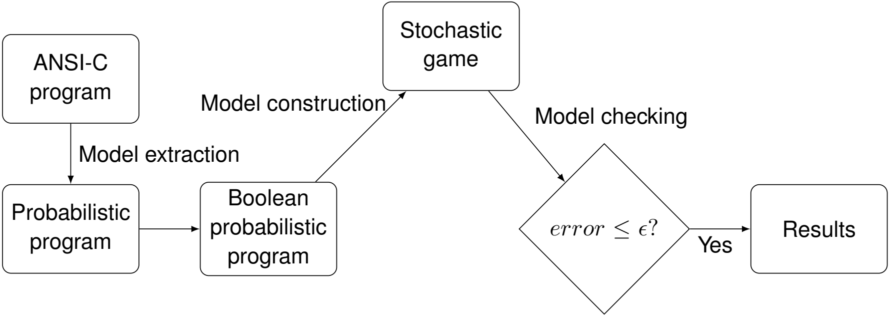


Outline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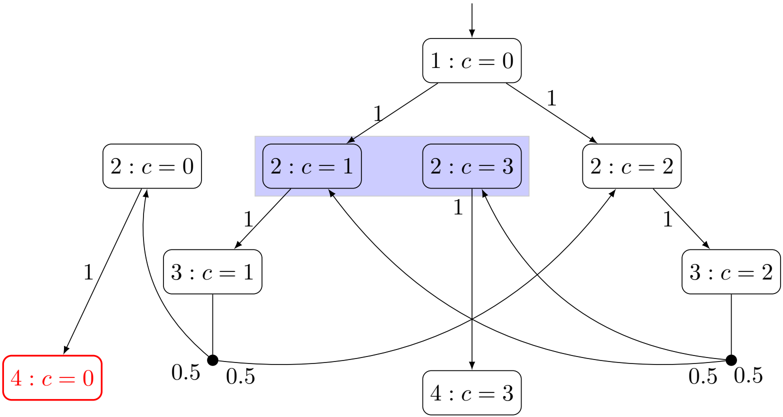
Stochastic game
Model checking
$error \leq \epsilon?$
Yes
Results

Abstraction of Gambler's ruin

Predicates: $c = 0, c = 2$



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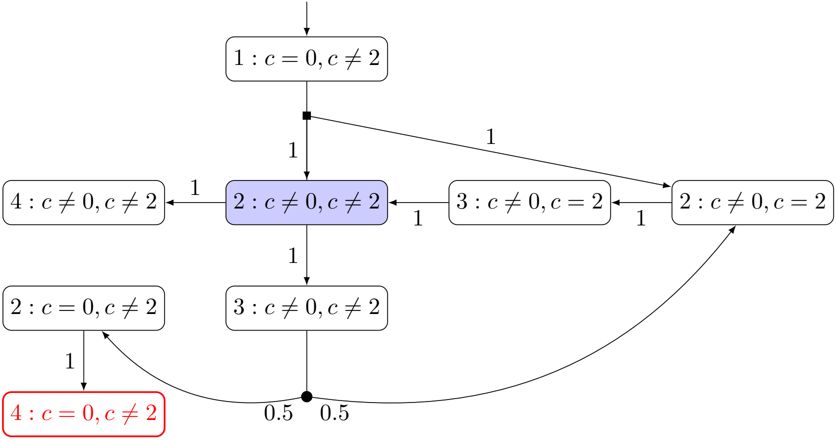

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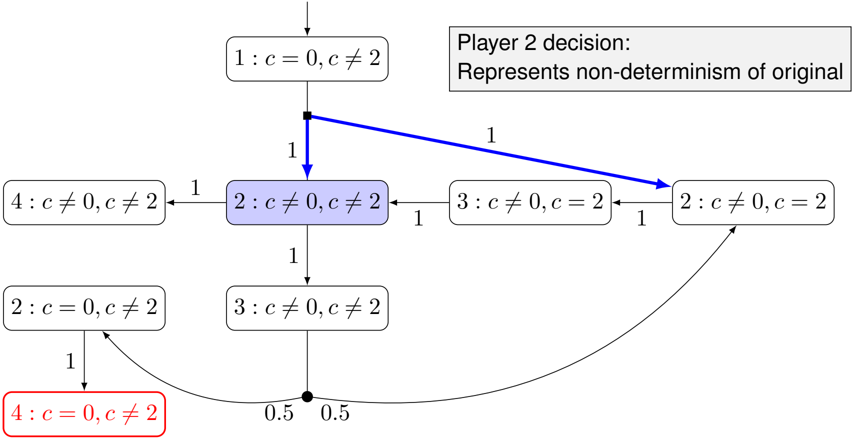

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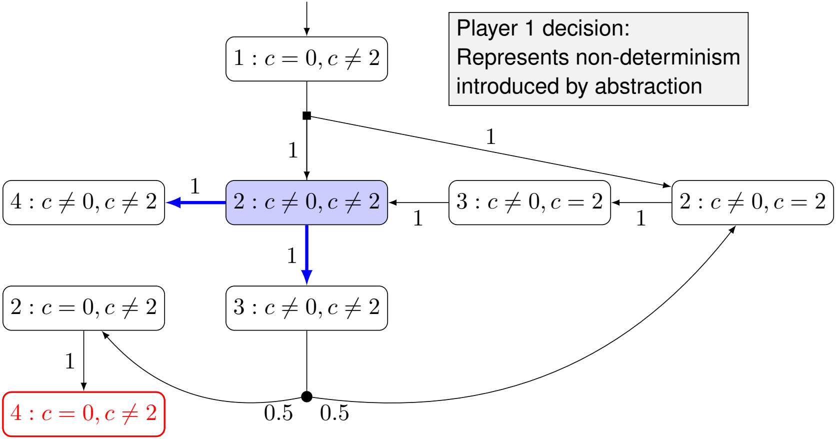
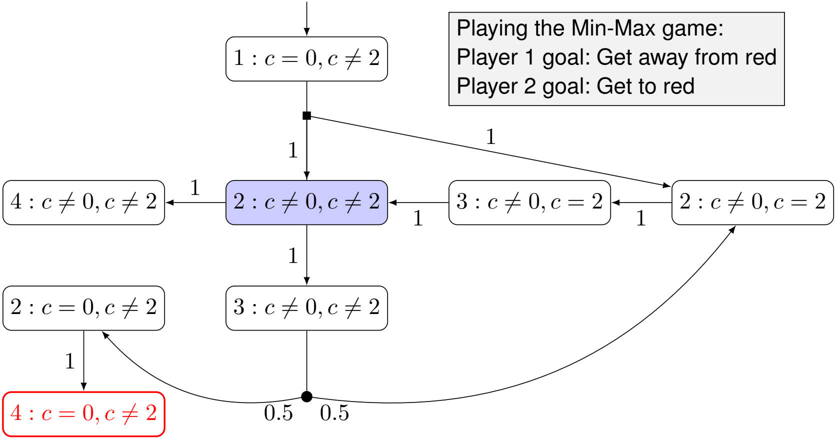


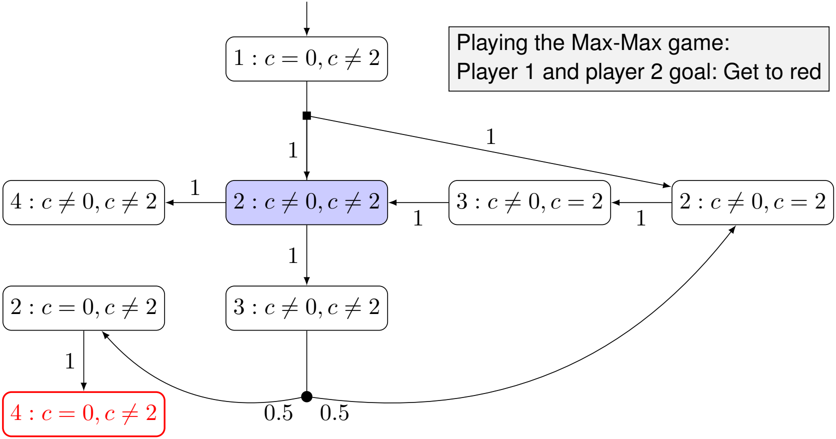
Abstraction of Gambler's ruin

Playing the Min-Max game:
Player 1 goal: Get away from red
Player 2 goal: Get to red



Abstraction of Gambler's ruin

Playing the Max-Max game:
Player 1 and player 2 goal: Get to red



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Model checking

Theorem

Probabilities in abstraction give bounds on original probabilities

$$\mathbf{p}^{Min-Min}(\hat{\mathcal{F}}) \leq \mathbf{p}^{Min}(\mathcal{F}) \leq \mathbf{p}^{Max-Min}(\hat{\mathcal{F}})$$

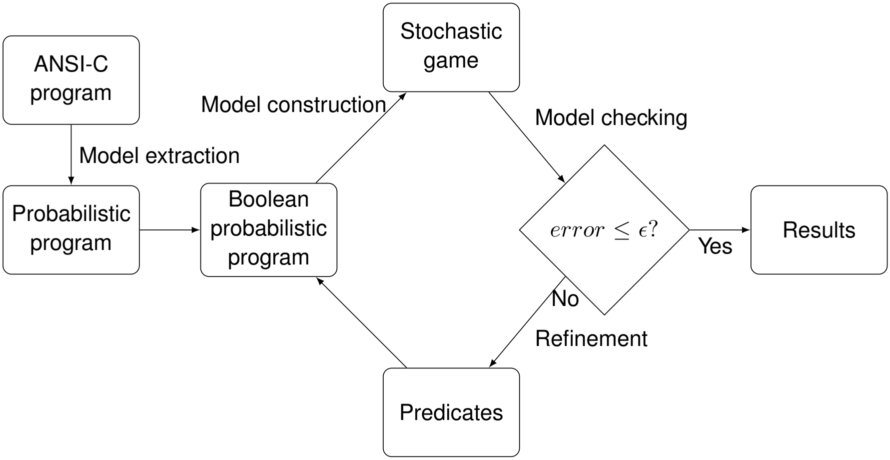
$$\mathbf{p}^{Min-Max}(\hat{\mathcal{F}}) \leq \mathbf{p}^{Max}(\mathcal{F}) \leq \mathbf{p}^{Max-Max}(\hat{\mathcal{F}})$$

$$\mathbf{p}^{Min-Max}(\Diamond \text{red}) = 0 \leq \mathbf{p}^{Max}(\Diamond \text{red}) \leq 1 = \mathbf{p}^{Max-Max}(\Diamond \text{red})$$

Problem

No meaningful information: error $\epsilon = 1 \implies$ Imprecise abstraction

Outline



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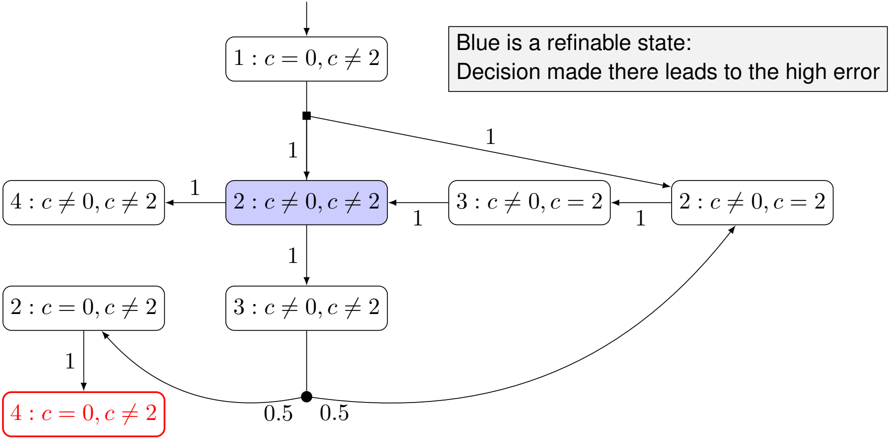
$error \leq \epsilon?$
Yes
Results
No
Refinement
Predicates

Abstraction refinement

Solution: Abstraction refinement

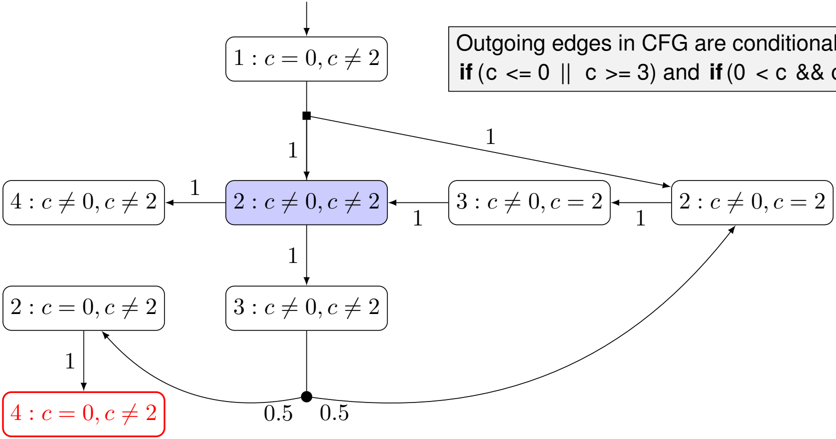
- Identify refinable state with distinct choices in min and max case, e.g.
  - state with highest error
    (difference between minimal and maximal probability)
  - state nearest to initial state
- Case distinction on the label of the outgoing edges

Abstraction of Gambler's ruin



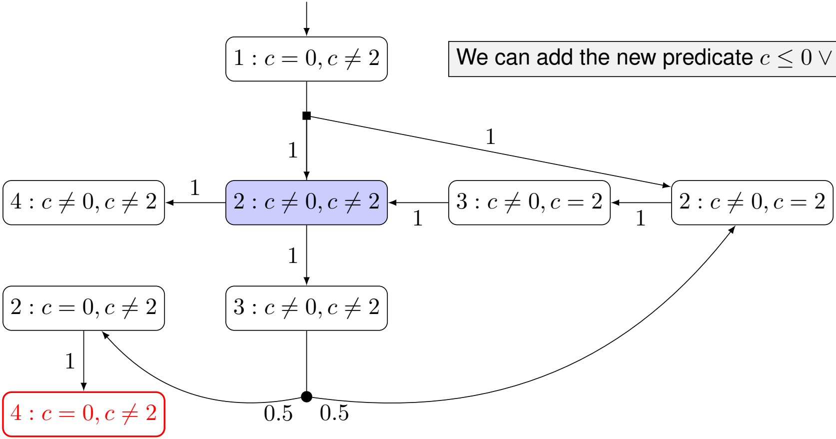
Abstraction of Gambler's ruin

Outgoing edges in CFG are conditionals
if (c <= 0 || c >= 3) and **if** (0 < c && c < 3)



Abstraction of Gambler's ruin

We can add the new predicate $c \leq 0 \vee c \geq 3$



Abstraction of Gambler's ruin

- Model checking the abstraction gives exact results (error $\epsilon = 0$)
- Refined abstraction is isomorphic to the original program
  Using the predicates we can separate all possible values of c
- So here we haven't gained much
- But the method is very efficient for real-world examples

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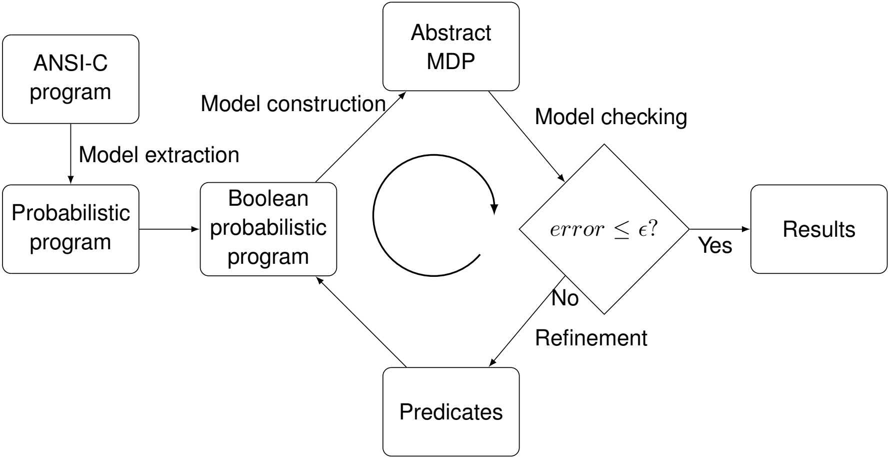
Abstraction-refinement loop



Thank you for your attention

Questions?