Semantics and Verification of Software

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Lecture 11: Axiomatic Semantics of WHILE III (Total Correctness)

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Hoare Logic

Goal: syntactic derivation of valid partial correctness properties. Here $A[x \mapsto a]$ denotes the syntactic replacement of every occurrence of x by a in A.

Tony Hoare (* 1934)

Definition (Hoare Logic)

The Hoare rules are given by

$$\text{(skip)}\ \frac{}{\{A\}\ \texttt{skip}\ \{A\}} \qquad \text{(asgn)}\ \frac{}{\{A[x \mapsto a]\}\ x\,{:=}\,a\ \{A\}}$$

$$\text{(seq)}\ \frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1\,;c_2\ \{B\}} \qquad \text{(if)}\ \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}\ \{B\}}$$

$$\text{(while)}\ \frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}\ \{A \wedge \neg b\}} \qquad \text{(cons)}\ \frac{\models (A \Rightarrow A') \quad \{A'\}\ c\ \{B'\} \quad \models (B' \Rightarrow B)}{\{A\}\ c\ \{B\}}$$

A partial correctness property is provable (notation: $\vdash \{A\}\ c\ \{B\}$) if it is derivable by the Hoare rules. In (while), A is called a (loop) invariant.

Soundness of Hoare Logic

Theorem (Soundness of Hoare Logic)

For every partial correctness property $\{A\}\,c\,\{B\}$,

$$\vdash \{A\}\,c\,\{B\} \quad \Rightarrow \quad \models \{A\}\,c\,\{B\}.$$

Proof.

Let $\vdash \{A\}\,c\,\{B\}$. By induction over the structure of the corresponding proof tree we show that, for every $\sigma \in \Sigma$ and $I \in \mathit{Int}$ such that $\sigma \models^I A$, $\mathfrak{C}[\![c]\!]\sigma \models^I B$ (on the board). (If $\sigma = \bot$, then $\mathfrak{C}[\![c]\!]\sigma = \bot \models^I B$ holds trivially.) □

Incompleteness of Hoare Logic I

Soundness: only valid partial correctness properties are provable ✓

Completeness: all valid partial correctness properties are systematically derivable ↯

Theorem (Gödel's Incompleteness Theorem)

The set of all valid assertions

$$\{A \in Assn \mid \models A\}$$

is not recursively enumerable, i.e., there exists no proof system for Assn in which all valid assertions are systematically derivable.

Proof.

see [Winskel 1996, p. 110 ff] □

Kurt Gödel
(1906–1978)

Incompleteness of Hoare Logic II

Corollary

There is no proof system in which all valid partial correctness properties can be enumerated.

Proof.

Given $A \in Assn$, $\models A$ is obviously equivalent to $\{\mathsf{true}\}\ \texttt{skip}\ \{A\}$. Thus the enumerability of all valid partial correctness properties would imply the enumerability of all valid assertions. □

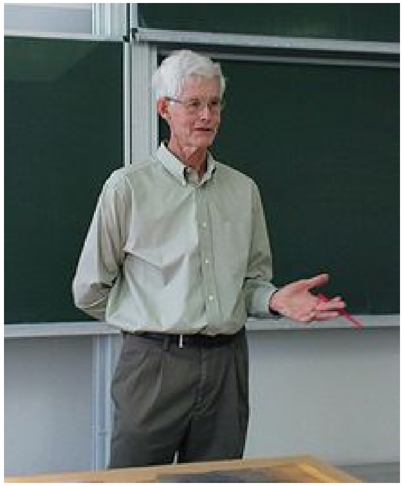
Remark: alternative proof (using computability theory):
$\{\mathsf{true}\}\ c\ \{\mathsf{false}\}$ is valid iff c does not terminate on any input state. But the set of all non-terminating WHILE statements is not enumerable.

Relative Completeness of Hoare Logic II

Theorem (Cook's Completeness Theorem)

Hoare Logic is relatively complete, i.e., for every partial correctness property $\{A\}\, c\, \{B\}$*:*

$$\models \{A\}\, c\, \{B\} \quad \Rightarrow \quad \vdash \{A\}\, c\, \{B\}.$$



Stephen A. Cook (* 1939)

Thus: if we know that a partial correctness property is valid, then we know that there is a corresponding derivation.

The proof uses the following concept: assume that, e.g., $\{A\}\, c_1 \,;\, c_2\, \{B\}$ has to be derived. This requires an intermediate assertion $C \in Assn$ such that $\{A\}\, c_1\, \{C\}$ and $\{C\}\, c_2\, \{B\}$. How to find it?

Total Correctness

- **Observation:** partial correctness properties only speak about terminating computations of a given program
- Total correctness additionally requires the proof that the program indeed stops (on the input states admitted by the precondition)
- Consider total correctness properties of the form

$$\{A\}\ c\ \{\Downarrow B\}$$

  where $c \in Cmd$ and $A, B \in Assn$
- Interpretation:

Validity of property $\{A\}\ c\ \{\Downarrow B\}$

For all states $\sigma \in \Sigma$ which satisfy A:
the execution of c in σ terminates and yields a state which satisfies B.

Semantics of Total Correctness Properties

Definition 11.1 (Semantics of total correctness properties)

Let $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$.

- $\{A\}\, c\, \{\Downarrow B\}$ is called valid in $\sigma \in \Sigma$ and $I \in \mathit{Int}$ (notation: $\sigma \models^I \{A\}\, c\, \{\Downarrow B\}$) if $\sigma \models^I A$ implies that $\mathfrak{C}[\![c]\!]\sigma \neq \bot$ and $\mathfrak{C}[\![c]\!]\sigma \models^I B$.
- $\{A\}\, c\, \{\Downarrow B\}$ is called valid in $I \in \mathit{Int}$ (notation: $\models^I \{A\}\, c\, \{\Downarrow B\}$) if $\sigma \models^I \{A\}\, c\, \{\Downarrow B\}$ for every $\sigma \in \Sigma$.
- $\{A\}\, c\, \{\Downarrow B\}$ is called valid (notation: $\models \{A\}\, c\, \{\Downarrow B\}$) if $\models^I \{A\}\, c\, \{\Downarrow B\}$ for every $I \in \mathit{Int}$.

Obviously, total implies partial correctness (but not vice versa):

Corollary 11.2

For all $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$,

$$\models \{A\}\, c\, \{\Downarrow B\} \quad \Rightarrow \quad \models \{A\}\, c\, \{B\}.$$

Proving Total Correctness I

Goal: syntactic derivation of valid total correctness properties

Definition 11.3 (Hoare Logic for total correctness)

The Hoare rules for total correctness are given by (where $i \in LVar$)

$$\text{(skip)}\ \frac{}{\{A\}\ \texttt{skip}\ \{\Downarrow A\}} \qquad \text{(asgn)}\ \frac{}{\{A[x \mapsto a]\}\ x := a\ \{\Downarrow A\}}$$

$$\text{(seq)}\ \frac{\{A\}\ c_1\ \{\Downarrow C\} \quad \{C\}\ c_2\ \{\Downarrow B\}}{\{A\}\ c_1 ; c_2\ \{\Downarrow B\}} \qquad \text{(if)}\ \frac{\{A \wedge b\}\ c_1\ \{\Downarrow B\} \quad \{A \wedge \neg b\}\ c_2\ \{\Downarrow B\}}{\{A\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}\ \{\Downarrow B\}}$$

$$\text{(while)}\ \frac{\models (i \geq 0 \wedge A(i+1) \Rightarrow b) \quad \{i \geq 0 \wedge A(i+1)\}\ c\ \{\Downarrow A(i)\} \quad \models (A(0) \Rightarrow \neg b)}{\{\exists i. i \geq 0 \wedge A(i)\}\ \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}\ \{\Downarrow A(0)\}}$$

$$\text{(cons)}\ \frac{\models (A \Rightarrow A') \quad \{A'\}\ c\ \{\Downarrow B'\} \quad \models (B' \Rightarrow B)}{\{A\}\ c\ \{\Downarrow B\}}$$

A total correctness property is provable (notation: $\vdash \{A\}\ c\ \{\Downarrow B\}$) if it is derivable by the Hoare rules. In case of (while), $A(i)$ is called a (loop) invariant.

Proving Total Correctness II

- In rule

$$(\text{while})\ \frac{\models (i \geq 0 \land A(i+1) \Rightarrow b) \quad \{i \geq 0 \land A(i+1)\}\, c\, \{\Downarrow A(i)\} \quad \models (A(0) \Rightarrow \neg b)}{\{\exists i. i \geq 0 \land A(i)\}\ \texttt{while}\ b\ \texttt{do}\ c\ \texttt{end}\ \{\Downarrow A(0)\}}$$

  the notation $A(i)$ indicates that assertion A parametrically depends on the value of the logical variable $i \in LVar$.
- Idea: i represents the remaining number of loop iterations
- Loop to be traversed $i + 1$ times ($i \geq 0$)
  $\Rightarrow A(i+1)$ holds
  $\Rightarrow$ execution condition b satisfied

  Thus: $\models (i \geq 0 \land A(i+1) \Rightarrow b)$, and $i + 1$ decreased to i after execution of c
- Execution terminated
  $\Rightarrow A(0)$ holds
  $\Rightarrow$ execution condition b violated

  Thus: $\models (A(0) \Rightarrow \neg b)$

Total Correctness of Factorial Program I

Example 11.4

Proof of $\{A\}$ `y:=1;` c $\{\Downarrow B\}$ where

$$A := (\mathtt{x} > 0 \land \mathtt{x} = i)$$
$$c := \mathtt{while}\ \neg(\mathtt{x=1})\ \mathtt{do}\ \mathtt{y:=y*x;}\ \mathtt{x:=x-1}\ \mathtt{end}$$
$$B := (\mathtt{y} = i!)$$

First we show that the assertion $C(j) = (\mathtt{x} > 0 \land \mathtt{y} * \mathtt{x}! = i! \land \mathtt{x} = j + 1)$ is an invariant of c. Applying (asgn) twice yields

$$\vdash \{j \geq 0 \land C(j)[\mathtt{x} \mapsto \mathtt{x-1}]\}\ \mathtt{x:=x-1}\ \{\Downarrow j \geq 0 \land C(j)\} \quad \text{and}$$
$$\vdash \{j \geq 0 \land C(j)[\mathtt{x} \mapsto \mathtt{x-1}][\mathtt{y} \mapsto \mathtt{y*x}]\}\ \mathtt{y:=y*x}\ \{\Downarrow j \geq 0 \land C(j)[\mathtt{x} \mapsto \mathtt{x-1}]\}$$

such that (seq) implies

$$\vdash \{j \geq 0 \land C(j)[\mathtt{x} \mapsto \mathtt{x-1}][\mathtt{y} \mapsto \mathtt{y*x}]\}\ \mathtt{y:=y*x;}\ \ \mathtt{x:=x-1}\ \{\Downarrow j \geq 0 \land C(j)\}.$$

Now $C(j+1) = (\mathtt{x} > 0 \land \mathtt{y*x}! = i! \land \mathtt{x} = j + 2)$ and
$C(j)[\mathtt{x} \mapsto \mathtt{x-1}][\mathtt{y} \mapsto \mathtt{y*x}] = (\mathtt{x} - 1 > 0 \land \mathtt{y} * \mathtt{x} * (\mathtt{x} - 1)! = i! \land \mathtt{x} - 1 = j + 1)$
such that

$$\models ((j \geq 0 \land C(j+1)) \Rightarrow (j \geq 0 \land C(j)[\mathtt{x} \mapsto \mathtt{x-1}][\mathtt{y} \mapsto \mathtt{y*x}])) \text{ and}$$
$$\models ((j \geq 0 \land C(j)) \Rightarrow C(j)).$$

Total Correctness of Factorial Program II

Example 11.4 (continued)

Hence (cons) implies

$$\vdash \{j \geq 0 \wedge C(j+1)\}\, \texttt{y:=y*x; x:=x-1}\, \{\Downarrow C(j)\}.$$

Moreover we have

$$\models ((j \geq 0 \wedge C(j+1)) \Rightarrow \neg(\mathtt{x} = 1)) \text{ and } \models (C(0) \Rightarrow \neg(\neg(\mathtt{x} = 1)))$$

such that (while) yields

$$\vdash \{\exists j. j \geq 0 \wedge C(j)\}\, c\, \{\Downarrow C(0)\}.$$

For the initializing assignment, (asgn) implies

$$\vdash \{\exists j. j \geq 0 \wedge C(j)[\mathtt{y} \mapsto 1]\}\, \texttt{y:=1}\, \{\Downarrow \exists j. j \geq 0 \wedge C(j)\},$$

such that (seq) allows to conclude

$$\vdash \{\exists j. j \geq 0 \wedge C(j)[\mathtt{y} \mapsto 1]\}\, \texttt{y:=1;}\, c\, \{\Downarrow C(0)\}.$$

On the other hand we have (choose $j := i - 1$):

$$\models ((\mathtt{x} > 0 \wedge x = i) \Rightarrow (\exists j. j \geq 0 \wedge C(j)[\mathtt{y} \mapsto 1])) \text{ and } \models (C(0) \Rightarrow \mathtt{y} = i!)$$

such that (cons) yields the desired result:

$$\vdash \{\mathtt{x} > 0 \wedge \mathtt{x} = i\}\, \texttt{y:=1;}\, c\, \{\Downarrow \mathtt{y} = i!\}.$$

Soundness and Completeness of Hoare Logic for Total Correctness

Soundness

In analogy to Theorem 10.2 we can show that the Hoare Logic for total correctness properties is also sound:

Theorem 11.5 (Soundness)

For every total correctness property $\{A\}\,c\,\{\Downarrow B\}$,

$$\vdash \{A\}\,c\,\{\Downarrow B\} \quad \Rightarrow \quad \models \{A\}\,c\,\{\Downarrow B\}.$$

Proof.

again by structural induction over the derivation tree of $\vdash \{A\}\,c\,\{\Downarrow B\}$
(here only (while) case; on the board) □

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Relative Completeness

Also the counterpart to Cook's Completeness Theorem 10.5 applies:

Theorem 11.6 (Completeness)

The Hoare Logic for total correctness properties is relatively complete, i.e., for every $\{A\}\, c\, \{\Downarrow B\}$*:*

$$\models \{A\}\, c\, \{\Downarrow B\} \quad \Rightarrow \quad \vdash \{A\}\, c\, \{\Downarrow B\}.$$

Proof.

omitted □