Exercise Sheet 4: Denotational Semantics

Due date: May 12th. You can hand in your solutions at the start of the exercise class.

Exercise 1 (Chain Complete Partial Orders) 30%

Determine whether each of the following statements is true or false. For true statements present a formal proof, and for false statements provide a counterexample.

(a) [7.5%] Every continuous function $f\colon (D_1, \sqsubseteq_1) \to (D_1, \sqsubseteq_2)$ between two CCPOs $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ is monotonic.

(b) [7.5%] Consider the partial order $(\mathbb{Q}, \leq)$ of the rational numbers ordered by the natural order in the reals. $(\mathbb{Q}, \leq)$ is chain complete.

(c) [7.5%] If $f\colon (D_1, \sqsubseteq_1) \to (D_1, \sqsubseteq_2)$ is a monotonic function between two CCPOs and $D \subseteq D_1$ is a chain, then $f(\bigsqcup D) \sqsubseteq_2 \bigsqcup f(D)$.

(d) [7.5%] Let $(D, \sqsubseteq)$ be a partial order and let $f\colon (D, \sqsubseteq) \to (D, \sqsubseteq)$ be monotonic. If p is the least element in D satisfying $f(p) \sqsubseteq p$, then p is a fixed point of f.

Exercise 2 (repeat–until Loops) 25%

(a) [10%] Define a transformer $F\colon (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)$ such that

$$\mathfrak{C}[\![\mathsf{repeat}\ c\ \mathsf{until}\ b]\!] = \mathsf{fix}(F)\ .$$

The transformer F is allowed to depend on the semantics only of c and b (i.e. $\mathfrak{B}[\![b]\!]$ and $\mathfrak{C}[\![c]\!]$). You cannot rely on the existence of while-loops within the language to define F.

(b) [5%] Use the definition provided in (a) to compute the transformer $\hat{F}\colon (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)$ whose least fixed point gives the semantics of program repeat skip until false. In other words, compute $\hat{F}$ such that

$$\mathfrak{C}[\![\mathsf{repeat\ skip\ until\ false}]\!] = \mathsf{fix}(\hat{F})\ .$$

(c) [10%] Show that $\mathsf{fix}(\hat{F}) = f_\emptyset$.

Exercise 3 (Closed Sets) 15%

A set $C \subseteq D$ is *closed* if and only if for each chain $G \subseteq C$, $\bigsqcup G \in C$. In the following, let $(D, \sqsubseteq)$ be a chain complete partial order and $f : D \to D$ be a continous function. Prove the following two statements.

(a) [7.5%] For each closed set $C \subseteq D$ with $f(x) \in C$ for each $x \in C$, we have $\mathsf{fix}(f) \in C$.

(b) [7.5%] $f(x) \sqsubseteq x$ implies $\mathsf{fix}(f) \sqsubseteq x$, $x \in D$.

Exercise 4 (Pointwise Ordering) 30%

Let $(D, \sqsubseteq)$ be a CCPO and define $(D \to D, \sqsubseteq')$ by setting

$$f_1 \sqsubseteq' f_2 \text{ if and only if } f_1(d) \sqsubseteq f_2(d) \text{ for all } d \in D.$$

(a) [10%] Show that $(D \to D, \sqsubseteq')$ is a CCPO.

(b) [20%] Show that fixpoints of chains are “continous”, i.e.

$$\mathsf{fix}(\bigsqcup' \mathcal{F}) = \bigsqcup \{\mathsf{fix}(f) \mid f \in \mathcal{F}\}$$

holds for all non-empty chains $\mathcal{F} \subseteq D \to D$ of continous functions.