Exercise Sheet 3: Compiler Correctness

Due date: May 6th. You can hand in your solutions at the start of the exercise class.

Exercise 1 (AM Semantics) **20%**

(a) [5%] Write a WHILE program c that performs integer division x/y of two positive natural numbers given by variables x, y and stores the result in a variable z.

(b) [10%] Translate your program into an AM program using the translation functions $\mathfrak{T}_a, \mathfrak{T}_b$ and $\mathfrak{T}_c$ presented in the lecture.

(c) [5%] Give a run of the AM program, i.e. a sequence of AM configurations, starting in the initial configuration $\mathfrak{T}_c[\![c]\!] \vdash \langle 0, \varepsilon, \sigma[x \mapsto 5, y \mapsto 3, z \mapsto 0]\rangle$.

Exercise 2 (Compiler Correctness) **50%**

In the lecture, you encountered Theorem 5.9 which intuitively states that every WHILE program is translated by the "compiler" $\mathfrak{T}_c$ into an equivalent AM program, i.e. for every $c \in \mathsf{Cmd}$, $\mathfrak{O}[\![c]\!] = \mathfrak{M}[\![\mathfrak{T}_c[\![c]\!]]\!]$.

(a) [10%] The following statement is known as the **decomposition lemma** for AM programs. Let $c_1, c_2 \in \mathsf{Cmd}$ and $pc \in \{0, ..., |\mathfrak{T}_c[\![c_1]\!]| - 1\}$. If $\mathfrak{T}_c[\![c_1]\!]; \mathfrak{T}_c[\![c_2]\!] \vdash \langle pc, e, \sigma\rangle \rhd^k \langle |\mathfrak{T}_c[\![c_1]\!]; \mathfrak{T}_c[\![c_2]\!]|, e'', \sigma''\rangle$, then there exists a configuration $\langle pc', e', \sigma'\rangle$ and natural numbers k_1, k_2 with $k = k_1 + k_2$ such that $\mathfrak{T}_c[\![c_1]\!] \vdash \langle pc, e, \sigma\rangle \rhd^{k_1} \langle |\mathfrak{T}_c[\![c_1]\!]|, e', \sigma'\rangle$ and $\mathfrak{T}_c[\![c_1]\!]; \mathfrak{T}_c[\![c_2]\!] \vdash \langle |\mathfrak{T}_c[\![c_1]\!]|, e', \sigma'\rangle \rhd^{k_2} \langle |\mathfrak{T}_c[\![c_1]\!]; \mathfrak{T}_c[\![c_2]\!]|, e'', \sigma''\rangle$.
Prove that the decomposition lemma is correct. You may use the following proposition without giving an explicit proof by structural induction:

$$\forall j \in \mathbb{N} : \mathfrak{T}_c[\![c]\!] \vdash \langle pc, e, \sigma\rangle \rhd^j \langle pc', e', \sigma'\rangle \text{ implies } pc' \in \{0, 1, ..., |\mathfrak{T}_c[\![c]\!]|\}, \quad (1)$$

where $pc \in \{0, ..., |\mathfrak{T}_c[\![c_1]\!]| - 1\}$.

(b) [35%] Complete the proof of Theorem 5.9 by proving the missing direction, i.e. show that for every $c \in \mathsf{Cmd}$, $\sigma, \sigma' \in \Sigma$ and $e \in \mathrm{Stk}$,

$$\mathfrak{T}_c[\![c]\!] \vdash \langle 0, \varepsilon, \sigma\rangle \rhd^\star \langle |\mathfrak{T}_c[\![c]\!]|, e, \sigma'\rangle \text{ implies } \langle c, \sigma\rangle \to \sigma' \text{ and } e = \varepsilon.$$

You may use all theorems and lemmata presented in the lecture so far (except Theorem 5.9 and Lemma 5.11 of course). In particular, note that Lemma 5.7 and Lemma 5.8 already entail the correctness of $\mathfrak{T}_a$ and $\mathfrak{T}_b$ in both directions, because the semantics of AM is deterministic and termination is guaranteed. Furthermore, you may assume that the decomposition lemma also works for arithmetic and Boolean expressions.

(c) [5%] Does the decomposition lemma also hold if $\mathfrak{T}_c[\![c_1]\!]$ and $\mathfrak{T}_c[\![c_2]\!]$ are replaced by arbitrary AM programs P_1 and P_2? Explain your answer.

Exercise 3 (Language Extension (again)) **30%**

(a) [10%] In the previous exercise sheet, you extended WHILE by an additional construct $\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b)$. Extend the translation function $\mathfrak{T}_c$ such that $\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b)$ statements can be translated directly, i.e. without rewriting them into normal WHILE programs first. For reasons of efficiency, we require the statement c to be translated only once.

(b) [20%] Prove that for all $\sigma, \sigma' \in \Sigma$, we have

$$\mathfrak{T}_c[\![\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b)]\!] \vdash \langle 0, \varepsilon, \sigma \rangle \rhd^\star \langle |\mathfrak{T}_c[\![\mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b)]\!]|, \varepsilon, \sigma' \rangle$$

if and only if $\langle \mathsf{repeat}\ \{c\}\ \mathsf{until}\ (b), \sigma \rangle \to \sigma'$.

You may use all theorems and lemmata presented in the lecture so far.