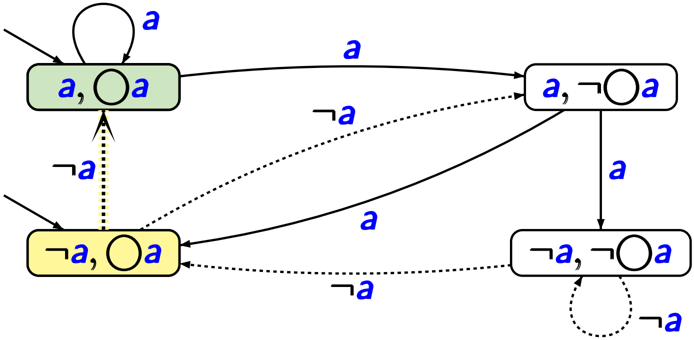
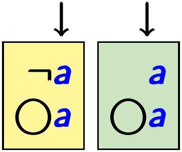
Example: GNBA for $\varphi = \bigcirc a$

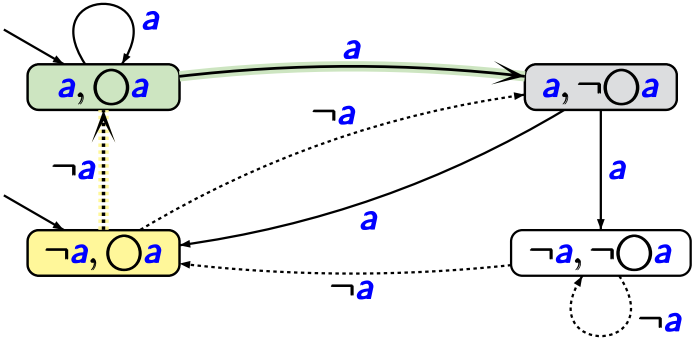


set of acceptance sets: $\mathcal{F} = \varnothing$

$\varnothing \quad \{a\} \quad \{a\} \quad \varnothing \quad \varnothing \quad \ldots \quad \models \bigcirc a$



Example: GNBA for $\varphi = \bigcirc a$



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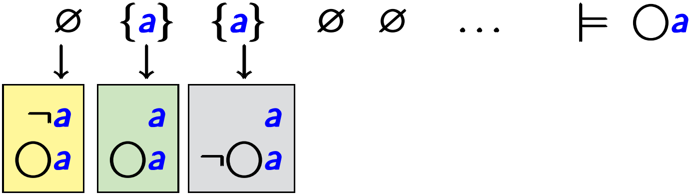

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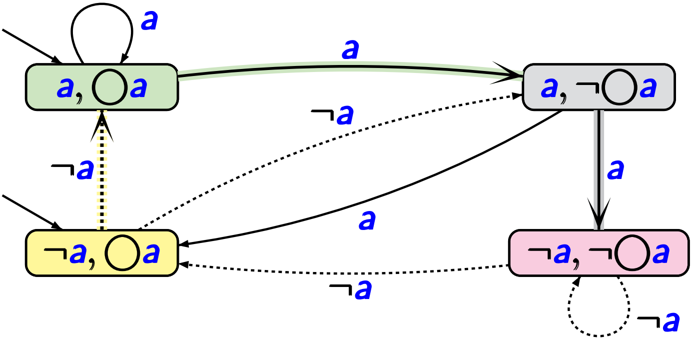

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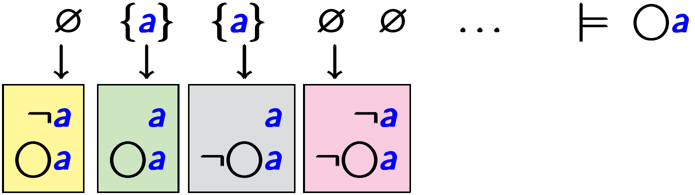

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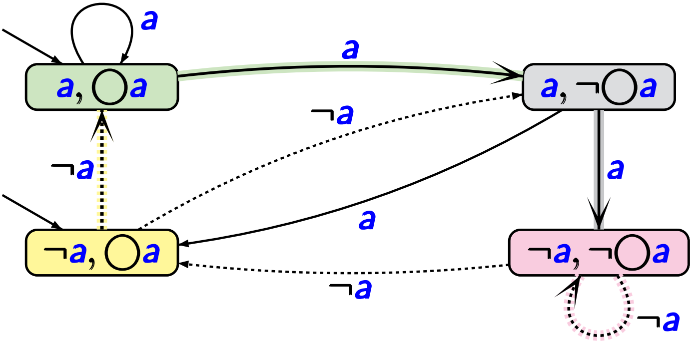

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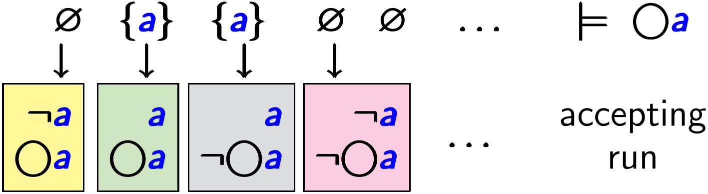
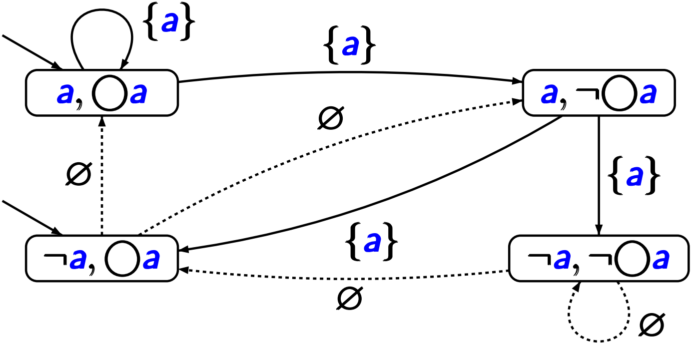
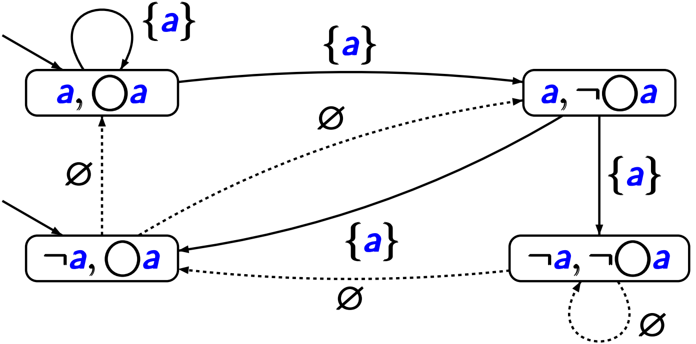


Soundness of the GNBA for $\varphi = \bigcirc a$



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

Soundness of the GNBA for $\varphi = \bigcirc a$



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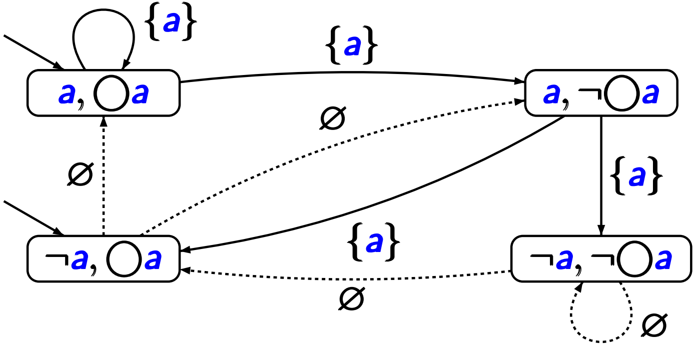
proof:

Soundness of the GNBA for $\varphi = \bigcirc a$



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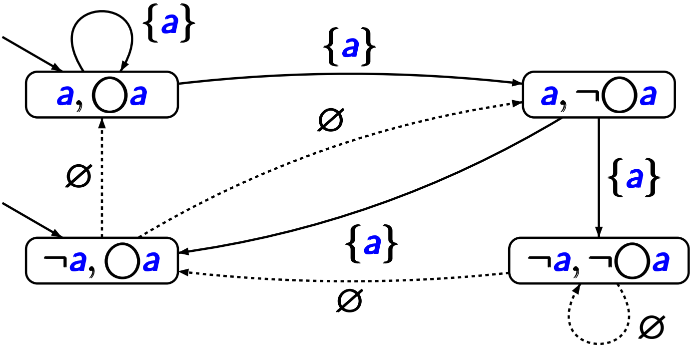

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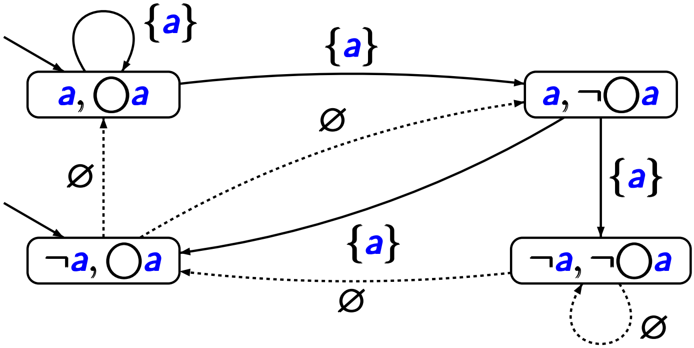

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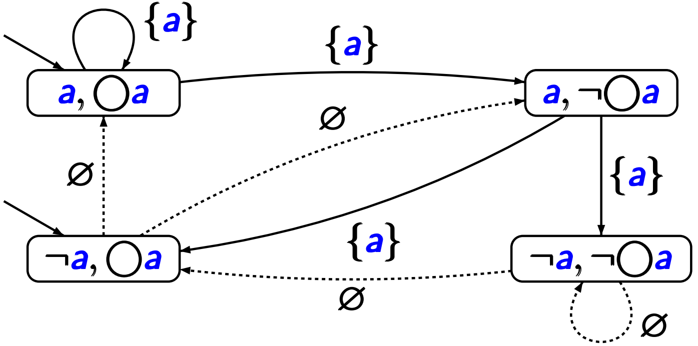

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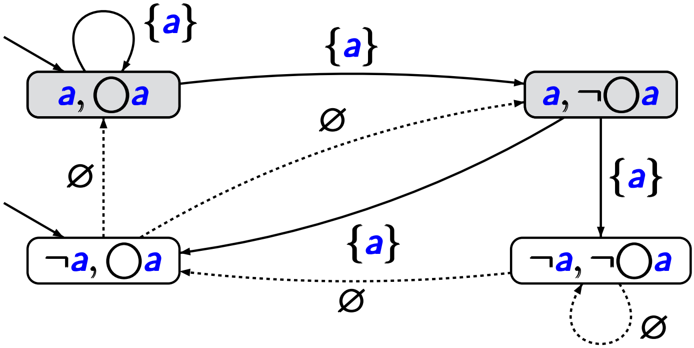

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$\Longrightarrow \quad$ the outgoing edges of B_1 have label $\{a\}$

$\Longrightarrow \quad \{a\} = B_1 \cap AP = A_1$

Example: GNBA for $\varphi = a \, \mathsf{U} \, b$

Example: GNBA for $\varphi = a \,\mathsf{U}\, b$

$a, b, a \,\mathsf{U}\, b$

$\neg a, \neg b, \neg(a \,\mathsf{U}\, b)$

$a, \neg b, a \,\mathsf{U}\, b$

$a, \neg b, \neg(a \,\mathsf{U}\, b)$

$\neg a, b, a \,\mathsf{U}\, b$

locally inconsistent: $\{a, b, \neg(a \,\mathsf{U}\, b)\}$

$\{\neg a, b, \neg(a \,\mathsf{U}\, b)\}$

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Example: GNBA for $\varphi = a \, \mathsf{U} \, b$

$a, b, a \, \mathsf{U} \, b$

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$a, \neg b, \neg(a \, \mathsf{U} \, b)$

$\neg a, b, a \, \mathsf{U} \, b$

initial states: B with $\varphi = a \, \mathsf{U} \, b \in B$

Example: GNBA for $\varphi = a \,\mathsf{U}\, b$

$\longrightarrow$ $a, b, a \,\mathsf{U}\, b$

$\neg a, \neg b, \neg(a \,\mathsf{U}\, b)$

$\longrightarrow$ $a, \neg b, a \,\mathsf{U}\, b$

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acceptance condition: just one set of accept states

F = set of all B with $\varphi \notin B$ or $b \in B$

Example: GNBA for $\varphi = a \, \mathsf{U} \, b$ ⟵ NBA

→ $a, b, a \, \mathsf{U} \, b$

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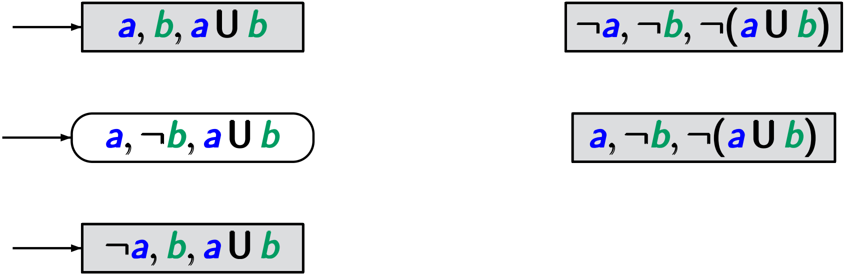
Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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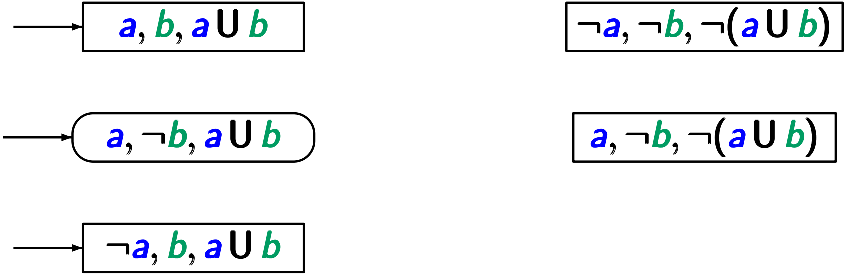
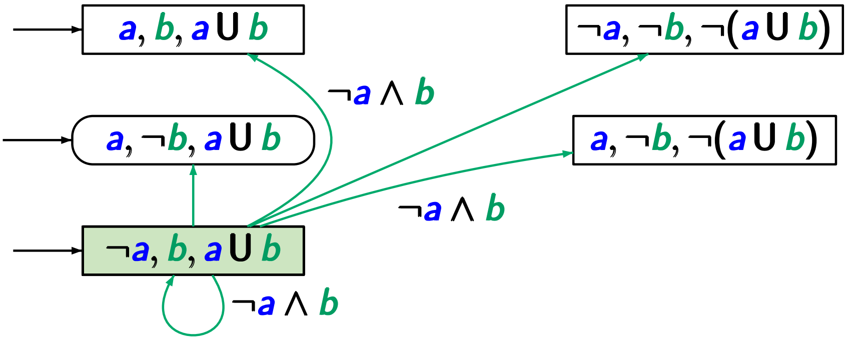


transition relation: $B' \in \delta(B, B \cap AP)$ iff

$$a \,\mathsf{U}\, b \in B \iff \big(b \in B \;\vee\; (\,a \in B \;\wedge\; a \,\mathsf{U}\, b \in B'\,)\big)$$

Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$



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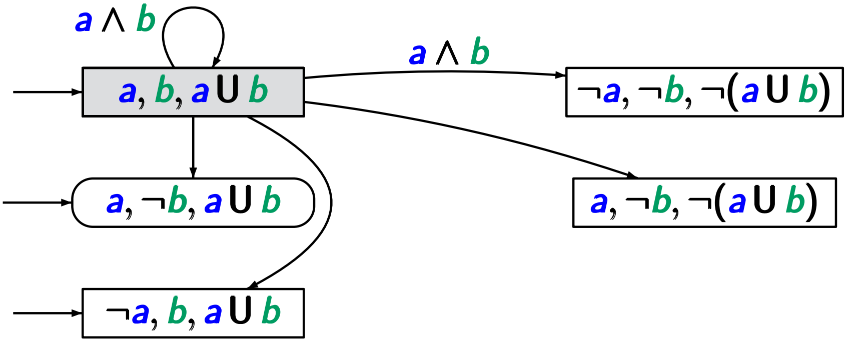

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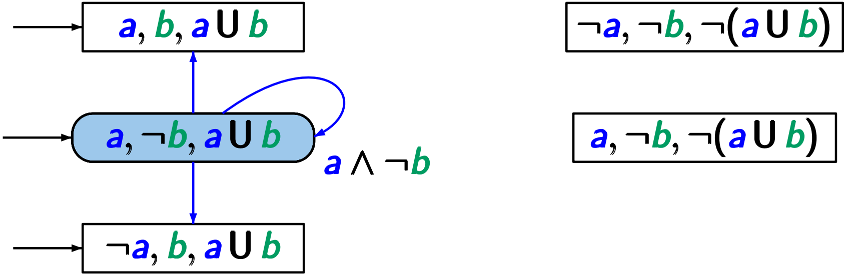

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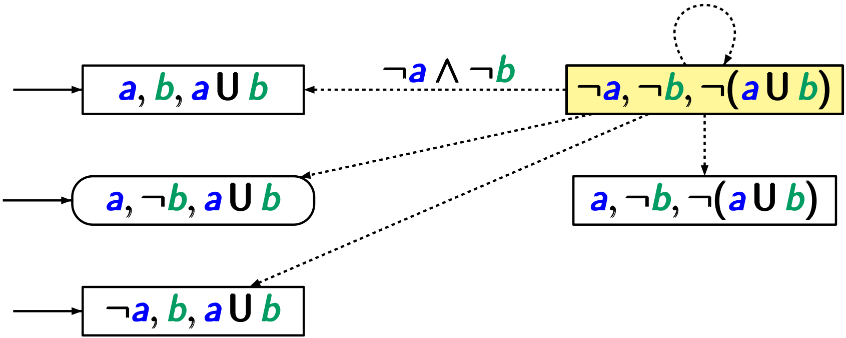

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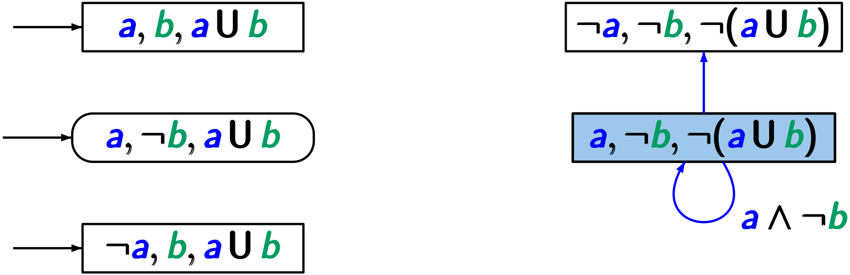
Example: (G)NBA for $\varphi = a \mathsf{U} b$



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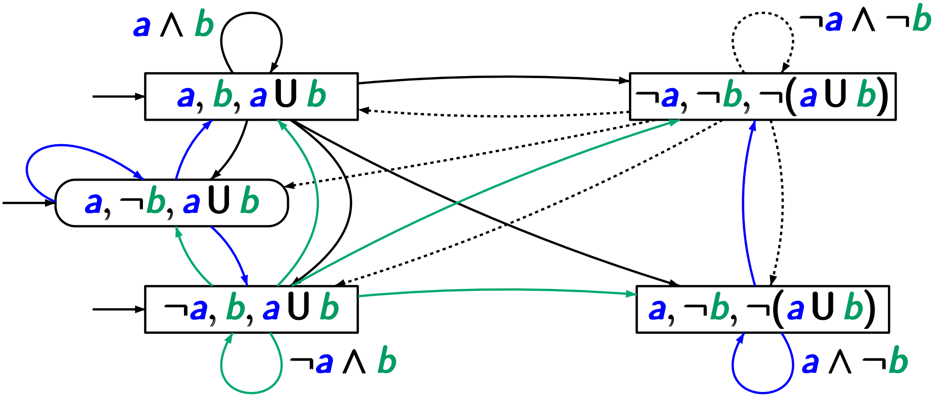
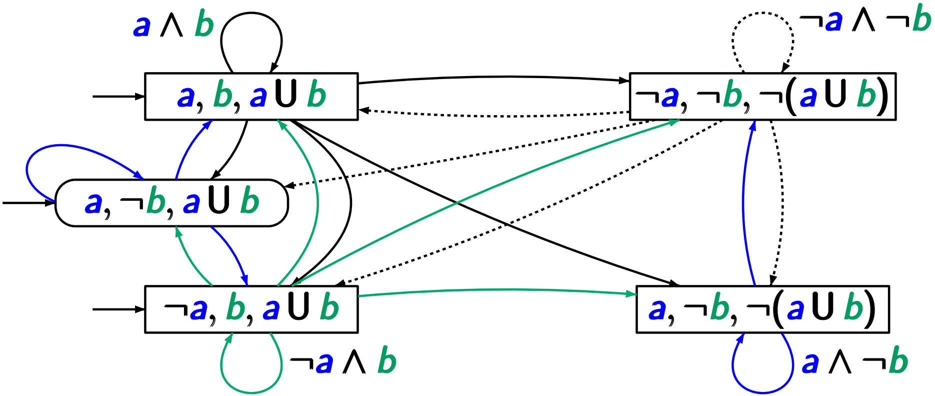
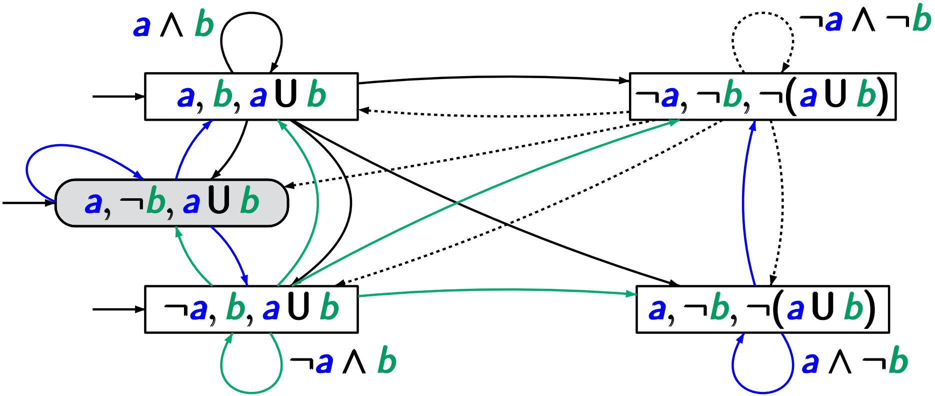


Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$



$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \, \mathsf{U} \, b$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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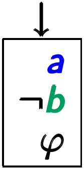

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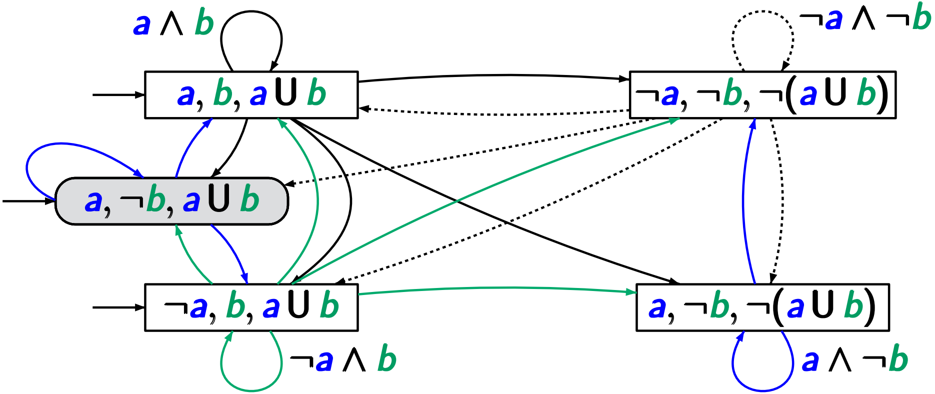

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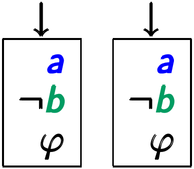
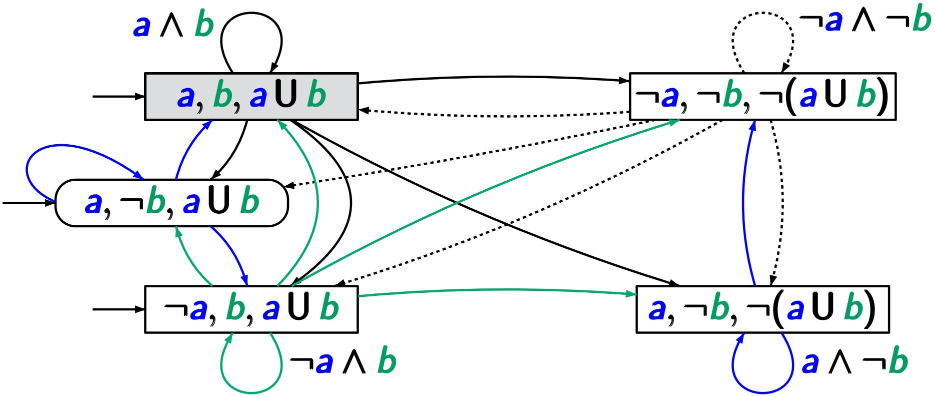


Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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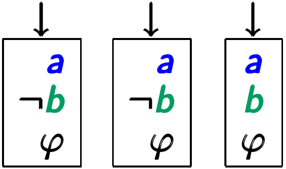

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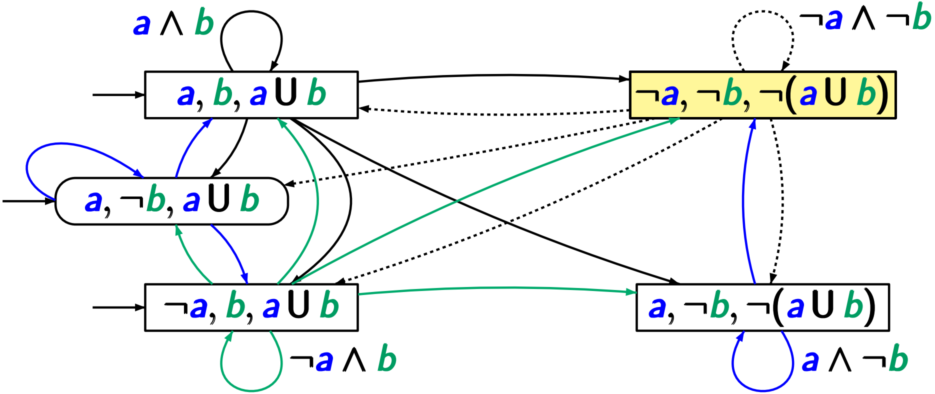

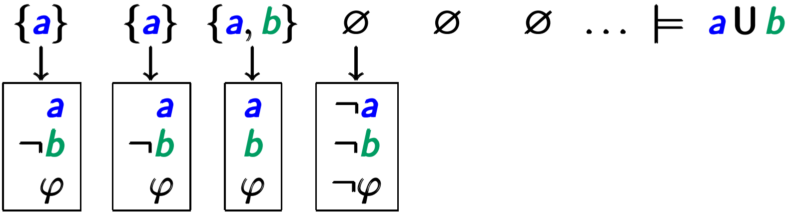

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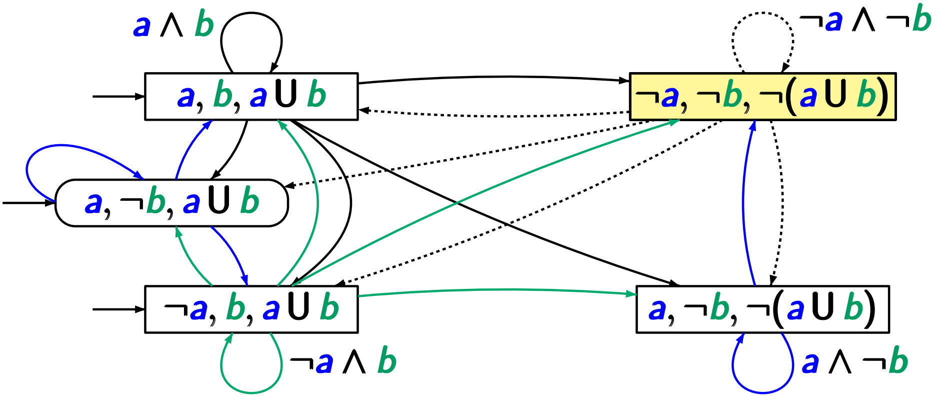
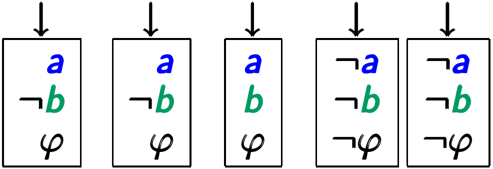


$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \,\mathsf{U}\, b$



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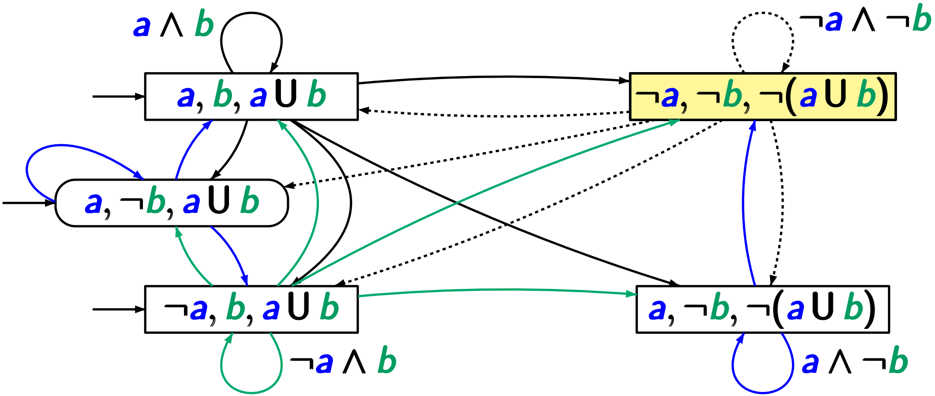
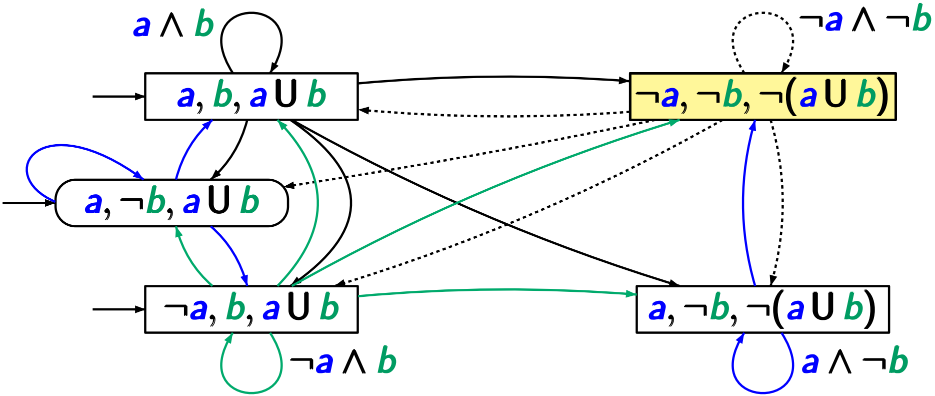


$$\{a\} \quad \{a\} \quad \{a,b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \,\mathsf{U}\, b$$

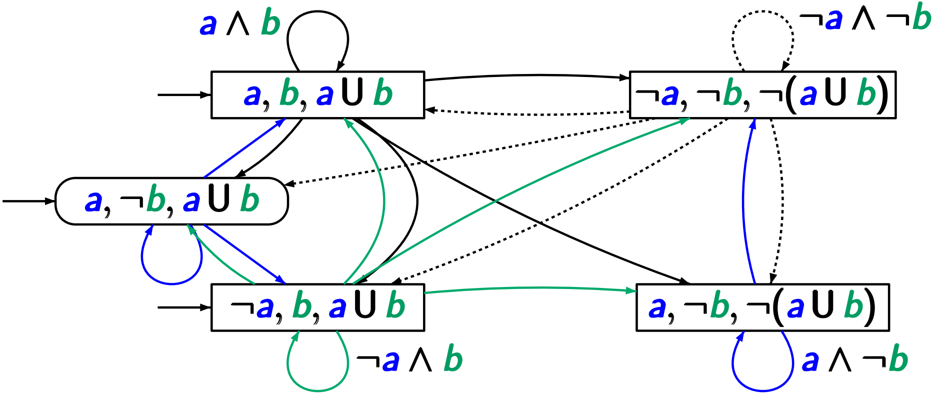
a	a	a	$\neg a$	$\neg a$	$\neg a$
$\neg b$	$\neg b$	b	$\neg b$	$\neg b$	$\neg b$
φ	φ	φ	$\neg\varphi$	$\neg\varphi$	$\neg\varphi$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



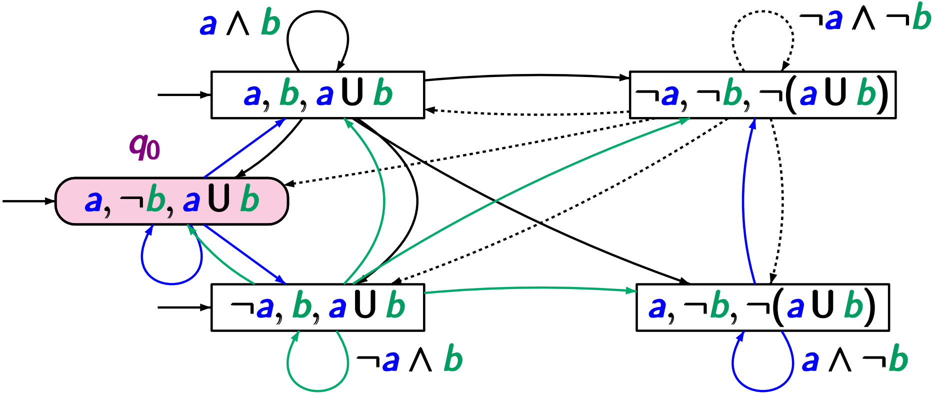
$\{a\}$	$\{a\}$	$\{a, b\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\ldots \models a \,\mathsf{U}\, b$
a, $\neg b$, φ	a, $\neg b$, φ	a, b, φ	$\neg a$, $\neg b$, $\neg\varphi$	$\neg a$, $\neg b$, $\neg\varphi$	$\neg a$, $\neg b$, $\neg\varphi$	accepting run

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



$$\{a\}\,\{a\}\,\{a\}\,\{a\}\,\ldots \not\models \varphi$$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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only **1** infinite run: $q_0\, q_0\, q_0\, \ldots$

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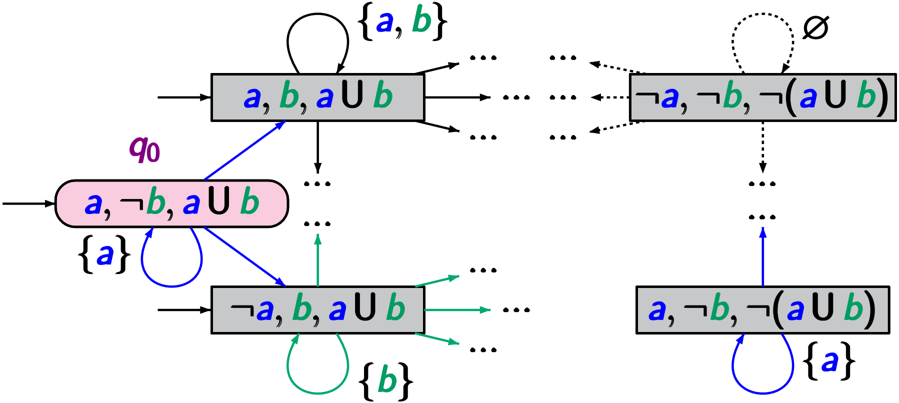

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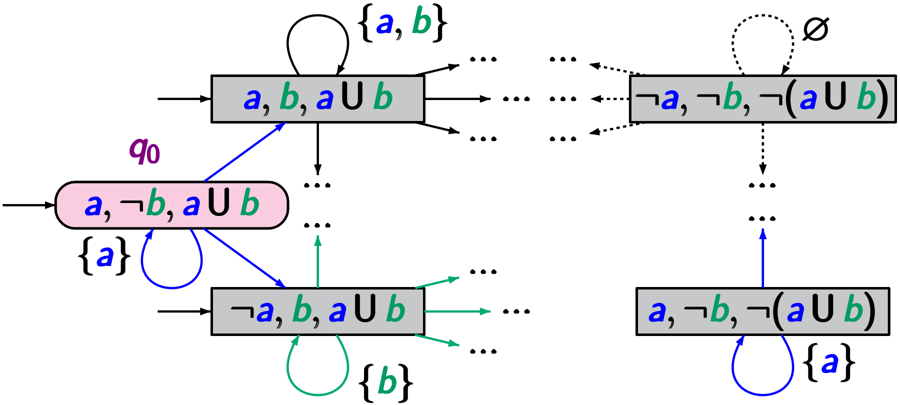

$\{a\}\,\{a\}\,\{a\}\,\{a\} \ldots \not\models \varphi$

only **1** infinite run: $q_0\, q_0\, q_0 \ldots$ not accepting

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Soundness

.... of the construction LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\, A_1\, A_2 \ldots \in (2^{AP})^\omega$

with $A_0\, A_1\, A_2 \ldots \models \varphi$

has an accepting run in $\mathcal{G}$

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\, A_1\, A_2 \ldots \in (2^{AP})^\omega$

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has an accepting run in $\mathcal{G}$

"$\supseteq$" show: for all infinite words $A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

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Accepting runs for the elements of *Words*(φ)

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ elementary formula-sets $B \subseteq cl(\varphi)$

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Example: $\varphi \;=\; a\, \mathsf{U}(\neg a \wedge b)$

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Example: $\varphi \;=\; a\, \mathsf{U}(\neg a \wedge b)$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

Accepting runs for the elements of $Words(\varphi)$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

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$$\begin{array}{cccccccc} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are states in $\mathcal{G}$, i.e., elementary subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

Accepting runs for the elements of $Words(\varphi)$

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Example: $\varphi \;=\; a\, \mathsf{U} (\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \quad \models \varphi$$

$\downarrow$

$$\begin{array}{|r|} \hline a \\ \neg b \\ \neg \psi \\ \varphi \\ \hline \end{array}$$

Accepting runs for the elements of $Words(\varphi)$

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$$\downarrow \qquad \downarrow$$

$$\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} \quad \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}}$$

Accepting runs for the elements of $Words(\varphi)$

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$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$

$\{a\}$	$\{a\}$	$\{a, b\}$
$\downarrow$	$\downarrow$	$\downarrow$
a	a	a
$\neg b$	$\neg b$	b
$\neg \psi$	$\neg \psi$	$\neg \psi$
φ	φ	φ

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

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Example: $\varphi = a \mathsf{U} (\neg a \wedge b)$ $\psi = \neg a \wedge b$

$$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
a	a	a	$\neg a$
$\neg b$	$\neg b$	b	b
$\neg\psi$	$\neg\psi$	$\neg\psi$	ψ
φ	φ	φ	φ

Accepting runs for the elements of $Words(\varphi)$

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ	a, b, $\neg\psi$, φ	$\neg a$, b, ψ, φ	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$			

Accepting runs for the elements of $Words(\varphi)$

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$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
a, $\neg b$, $\neg\psi$, φ	a, $\neg b$, $\neg\psi$, φ	a, b, $\neg\psi$, φ	$\neg a$, b, ψ, φ	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\neg a$, $\neg b$, $\neg\psi$, $\neg\varphi$	$\ldots$	

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \mathit{true} \in cl(\varphi) & \text{implies} & \mathit{true} \in B \end{array}$$

(ii) B is locally consistent with respect to until U,
i.e., if $\psi_1 \, \mathsf{U} \, \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{l} \text{if } \psi_1 \, \mathsf{U} \, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \, \mathsf{U} \, \psi_2 \in B \end{array}$$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0 A_1 A_2 \ldots \in (2^{AP})^\omega$

with $A_0 A_1 A_2 \ldots \models \varphi$

has an accepting run in $\mathcal{G}$

"$\supseteq$" show: for all infinite words $A_0 A_1 A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

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Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0. \ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 \, A_1 \, A_2 \, \ldots \models \psi$$

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as $B_0 \in Q_0$

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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and $(*)$ holds

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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$\Longrightarrow$ $\sigma = A_0\, A_1\, A_2 \ldots \models \varphi$

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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↑ as $B_0 \in Q_0$

$\Longrightarrow$ $\sigma = A_0\, A_1\, A_2 \ldots \models \varphi$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Proof by structural induction on ψ

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

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Proof by structural induction on ψ

base of induction:

$\psi = \textbf{\textit{true}}$

$\psi = a \in AP$

Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 \, A_1 \, A_2 \ldots \models \psi$$

Proof by structural induction on ψ

base of induction:

$\psi = \textit{true}$

$\psi = a \in AP$

induction step:

$\psi = \neg\psi'$

$\psi = \psi_1 \wedge \psi_2$

$\psi = \bigcirc\psi'$

$\psi = \psi_1 \, \mathsf{U} \, \psi_2$

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

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Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$.

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$

note: ***true*** is contained in all elementary formula-sets

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and $A_0\, A_1\, A_2 \ldots \models \textit{true}$

note: ***true*** is contained in all elementary formula-sets
true holds for all paths/traces

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

$$A_0\, A_1\, A_2 \ldots \models \textit{true}$$

Let $\psi = a \in AP$.

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Base of induction:

Suppose $\psi = \mathit{true} \in cl(\varphi)$. Then $\mathit{true} \in B_0$ and

$$A_0\, A_1\, A_2 \ldots \models \mathit{true}$$

Let $\psi = a \in AP$. Then:

$a \in B_0$

Base of induction

Claim: If $B_0 \stackrel{A_0}{\to} B_1 \stackrel{A_1}{\to} B_2 \stackrel{A_2}{\to} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \stackrel{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

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$$a \in B_0 \iff a \in A_0$$

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Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \neg\psi'$:

Induction step: negation

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

Induction step: negation

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Induction step: for $\psi = \neg\psi'$:

$\psi \in B_0$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \ldots \not\models \psi'$ (induction hypothesis)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \neg\psi'$:

$\psi \in B_0$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0\,A_1\,A_2 \ldots \not\models \psi'$ (induction hypothesis)

iff $A_0\,A_1\,A_2 \ldots \models \psi$ (semantics of $\neg$)

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\psi \notin B \quad \text{iff} \quad \neg\psi \in B$$
$$\psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B$$
$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until U,
i.e., if $\psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)$ then:

$$\text{if } \psi_1 \,\mathsf{U}\, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B$$
$$\text{if } \psi_2 \in B \text{ then } \psi_1 \,\mathsf{U}\, \psi_2 \in B$$

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(ii) B is locally consistent with respect to until $\mathbf{U}$,
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if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$
if $\psi_2 \in B$ then $\psi_1 \,\mathbf{U}\, \psi_2 \in B$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

Claim: If $B_0 \overset{A_0}{\rightarrow} B_1 \overset{A_1}{\rightarrow} B_2 \overset{A_2}{\rightarrow} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0 A_1 A_2 \ldots \models \psi_1$ and $A_0 A_1 A_2 \ldots \models \psi_2$ (IH)

iff $A_0 A_1 A_2 \ldots \models \psi$ (semantics of $\wedge$)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$: